Concise Calculus II

Sudipta Mallik

April 8, 2024

To my students

Contents

Preface

There are lots of calculus books out there. Some of them are extensive but their unholy weights could be physically harmful if not mentally! I wanted to create a calculus book which is not physically harmful. Obviously it had to be concise. Therefore I used mostly my lecture notes to convert them to this book. In my Calculus II class I mention many additional things beside what is in this book. Teachers using this book could do the same especially when giving insights and emphasizing something. Students could use this book for a quick introduction to a calculus II topic and use other resources when needed.

This book is meant to be a concise treatment of standard calculus II topics taught in many American universities. An earlier version of this book can be found at https://learnmathonline.org. The work for this book was partially supported by the Elevating Excellence Grant from the President of Northern Arizona University.

Flagstaff, Arizona, USA — Sudipta Mallik
May, 2023

Acknowledgments

I would like to thank the following mathematicians for their generous contributions to this book:

Swarup Ghosh of Southwestern Oklahoma State University contributed many exercise problems and some corrections.

Victoria Vakarchuk of Northern Arizona University contributed some exercise problems.

David Deville of Northern Arizona University corrected some typos.

Chapter 1

Calculus I Review

In this chapter we review the basic concepts from Calculus I. Throughout this book we study real-valued functions of a real variable, i.e., functions whose domains and codomains are subsets of the set $\mathbb{R}$ of real numbers.

1.1 Limits and Continuity

Definition (Limit of a function)**.** Suppose a function f is defined on an open interval I containing a number a except possibly at a itself. The number L is the *limit of f as x approaches a*, denoted by

$$\lim_{x\to a} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.[1]

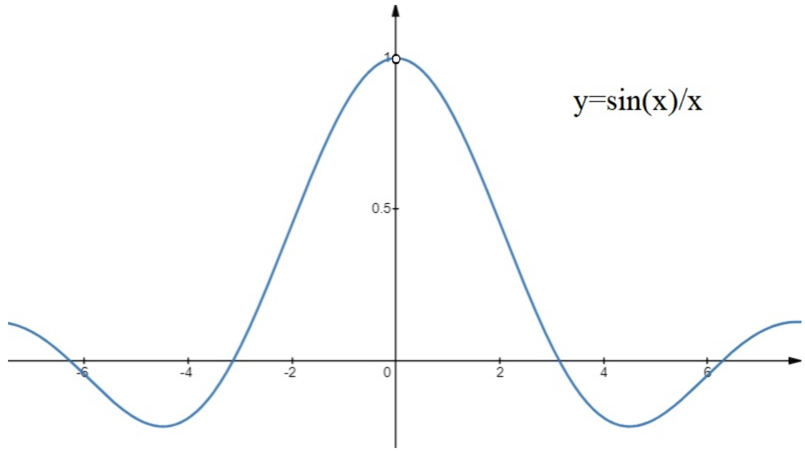
Example.

(a) The function $f(x) = \dfrac{\sin x}{x}$ is defined for all real number x except $x = 0$.

x	$f(x) = \dfrac{\sin x}{x}$
± 1	0.841470
± 0.1	0.998334
± 0.01	0.999983
± 0.001	0.999999

[1] "arbitrarily close" and "sufficiently close" can be made more clear by the following precise definition: The number L is the *limit of f as x approaches a* if for every positive ε, there is a positive δ such that

$$|f(x) - L| < \varepsilon \text{ for all } x \text{ satisfying } 0 < |x - a| < \delta.$$



From the graph of f and also from the function table $(x - f(x))$, we see that $f(x)$ can be made arbitrarily close to 1 by taking x sufficiently close to 0, i.e., $f(x)$ approaches 1 when x approaches 0 from the left and right. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(b) The function $f(x) = \sin\left(\frac{\pi}{x}\right)$ is defined for all real number x except $x = 0$. From the graph of f and also from the function table $(x - f(x))$, we see that $f(x)$ does not approach any fixed number L when x approaches 0. In particular, $f(x)$ oscillates between 1 and -1 as x approaches 0. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist.

Definition (Left-hand and right-hand limits of a function)**.** Suppose a function f is defined on an open interval I containing a number a except possibly at a itself.

(a) The number L is the *left-hand limit of f as x approaches a (from the left)*, denoted by

$$\lim_{x\to a^-} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a and keeping $x < a$.

(b) The number L is the *right-hand limit of f as x approaches a (from the right)*, denoted by

$$\lim_{x\to a^+} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a and keeping $x > a$.

Theorem 1.1.1. $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

Example.

1. From the graphical and numerical investigations in the last example, we see

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

and

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1 = \lim_{x\to 0^+} \frac{\sin x}{x}.$$

2. Consider the floor function f defined by $f(x) = \lfloor x \rfloor$ which is the largest integer less than or equal to x. Since $f(x) = 1$ for all x in $[1, 2)$ and $f(x) = 2$ for all x in $[2, 3)$,

$$\lim_{x\to 2^-} f(x) = 1 \text{ and } 2 = \lim_{x\to 2^+} f(x).$$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$ does not exist.

Theorem 1.1.2 (Algebra of limits)**.** *If* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist, then the following are true:*

(a) $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

(b) $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$ *for all real numbers* c.

(c) $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$.

(d) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *whenever* $\lim_{x\to a} g(x) \neq 0$.

(e) For any positive integer n,

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n \text{ and } \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},$$

assuming $\lim_{x\to a} f(x) \geq 0$ *for even* n *in the second limit.*

Example. Evaluate (a) $\lim_{x\to 2} \frac{8 - x\sqrt[3]{x^5 - 5}}{x^2 - 5x + 8}$, (b) $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$.

Solution. (a)

$$\begin{aligned}
\lim_{x\to 2}\frac{8-x\sqrt[3]{x^5-5}}{x^2-5x+8} &= \frac{\lim\limits_{x\to 2}\left(8-x\sqrt[3]{x^5-5}\right)}{\lim\limits_{x\to 2}\left(x^2-5x+8\right)}\\
&= \frac{\lim\limits_{x\to 2}8-\lim\limits_{x\to 2}x\sqrt[3]{x^5-5}}{\lim\limits_{x\to 2}x^2-\lim\limits_{x\to 2}5x+\lim\limits_{x\to 2}8}\\
&= \frac{8-\left(\lim\limits_{x\to 2}x\right)\left(\lim\limits_{x\to 2}\sqrt[3]{x^5-5}\right)}{\left(\lim\limits_{x\to 2}x\right)^2-5\left(\lim\limits_{x\to 2}x\right)+8}\\
&= \frac{8-2\sqrt[3]{\lim\limits_{x\to 2}\left(x^5-5\right)}}{2^2-5\cdot 2+8}\\
&= \frac{8-2\sqrt[3]{\left(\lim\limits_{x\to 2}x\right)^5-5}}{2}\\
&= \frac{8-2\sqrt[3]{\lim\limits_{x\to 2}\left(x^5-5\right)}}{2^2-5\cdot 2+8}\\
&= \frac{8-2\sqrt[3]{2^5-5}}{2}\\
&= 1
\end{aligned}$$

(b) First note that $f(x)=\dfrac{x^2-1}{x-1}$ is not defined at $x=1$. Since $\lim\limits_{x\to 1}(x-1)=0$, the quotient law is not applicable for the limit. When $x\to 1$, we see $x\neq 1$ and $x-1\neq 0$. Then we can evaluate the limit as follows:

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x+1)(x-1)}{(x-1)}\\
&= \lim_{x\to 1}(x+1) \text{ (Since } x-1\neq 0)\\
&= 2
\end{aligned}$$

Note that $y=x+1$ and $y=\dfrac{x^2-1}{x-1}$ have the same graph except at $x=1$.

Theorem 1.1.3. *Suppose $f(x)\leq g(x)$ for all x in an open interval containing a number a (except possibly a). If both $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ exist, then*

$$\lim_{x\to a}f(x)\leq\lim_{x\to a}g(x).$$

Theorem 1.1.4 (Sandwich/ Squeeze Theorem)**.** *Suppose $f(x)\leq g(x)\leq h(x)$ for all x in an open interval containing a number a (except possibly a). If $\lim\limits_{x\to a}f(x)=L=\lim\limits_{x\to a}h(x)$, then $\lim\limits_{x\to a}g(x)=L$.*

Example. Show that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

Solution. Note that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$. Then $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$ for all $x \ne 0$. Since $\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|$, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by the Sandwich Theorem.

Definition (Limit of a function at infinity)**.** Suppose a function f is defined on an open interval (a,∞) for some number a. The number L is the *limit of f as x approaches ∞*, denoted by

$$\lim_{x\to\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking sufficiently large positive x.[2] Similarly, for a function f defined on an open interval $(-\infty, a)$ for some number a, the number L is the *limit of f as x approaches $-\infty$*, denoted by

$$\lim_{x\to-\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking sufficiently large negative x. The line $y = L$ is a *horizontal asymptote* of the curve $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

Example.

1. From the graphical and numerical investigations, we see

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

   Therefore $y = 0$ is a horizontal asymptote of the curve $y = \frac{1}{x}$.

2. From the graphical and numerical investigations, we see

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}.$$

   Therefore $y = \frac{\pi}{2}$ is a horizontal asymptote of the curve $y = \tan^{-1} x$.

Remark. Algebra of limits are still valid for (finite) limits at infinity.

---

[2] "arbitrarily close" and "sufficiently large" can be made more clear by the following precise definition: The number L is the *limit of f as x approaches ∞* if for every positive ε, there is a positive N such that

$$|f(x) - L| < \varepsilon \text{ for all } x > N.$$

Definition (Infinite limit of a function)**.** We write

$$\lim_{x \to a} f(x) = \infty$$

if $f(x)$ can be made arbitrarily large positive number by taking x sufficiently close to a. Similarly

$$\lim_{x \to a} f(x) = -\infty$$

if $f(x)$ can be made arbitrarily large negative number by taking x sufficiently close to a. We have similar definitions for

$$\lim_{x \to a^+} f(x) = \infty, \ \lim_{x \to a^-} f(x) = \infty, \ \lim_{x \to a^+} f(x) = -\infty, \ \text{ and } \lim_{x \to a^-} f(x) = -\infty.$$

The line $x = a$ is a *vertical asymptote* of the curve $y = f(x)$ if at least one of the above limits exists.

Remark. Algebra of limits may not be valid for infinite limits.

Example. From the graphical and numerical investigations, we see

$$\lim_{x \to 0^+} \left(\ln x - \frac{1}{x} \right) = -\infty.$$

Then $x = 0$ is a vertical asymptote of $y = \ln x - \dfrac{1}{x}$. Note that the algebra of limits does not work as follows:

$$\lim_{x \to 0^+} \left(\ln x - \frac{1}{x} \right) = \lim_{x \to 0^+} \ln x - \lim_{x \to 0^+} \frac{1}{x} = -\infty - (-\infty).$$

Definition (Infinite limit of a function at infinity)**.** We write

$$\lim_{x \to \infty} f(x) = \infty$$

if $f(x)$ can be made arbitrarily large positive number by taking sufficiently large positive x. Similarly

$$\lim_{x \to \infty} f(x) = -\infty$$

if $f(x)$ can be made arbitrarily large negative number by taking sufficiently large positive x. We have similar definitions for

$$\lim_{x \to -\infty} f(x) = \infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

Example. From the graphical and numerical investigations, we see

1. $$\lim_{x \to \infty} e^x = \infty.$$

2.

$$\lim_{x\to-\infty} x^3 = -\infty.$$

Definition (Continuity)**.** A function f is *continuous at* a if $\lim_{x\to a} f(x) = f(a)$. We say f is *continuous on an open interval* if it is continuous at each number in the interval. A function f is *continuous* if it is continuous at each number of its domain. We say f is *discontinuous* at a number a in its domain if it is not continuous at a.

Example.

1. Consider the floor function f defined by $f(x) = \lfloor x \rfloor$ which is the largest integer less than or equal to x. For all integers n,

   $$\lim_{x\to n^-} f(x) = n - 1 \neq n = \lim_{x\to n^+} f(x).$$

   Then $\lim_{x\to n} f(x)$ does not exist and f is discontinuous at all integers n. But for all non-integer real numbers r,

   $$\lim_{x\to r} f(x) = \lfloor r \rfloor = f(r).$$

   Thus f is continuous at all non-integer real numbers r. Since f is not continuous at each number of its domain, f is not a continuous function.

2. $f(x) = \frac{1}{x}$ is a continuous function since it is continuous at each number of its domain. Note that 0 is not in the domain of $f(x) = \frac{1}{x}$.

Remark. The following functions are continuous: polynomials, rational functions, root functions, exponential and logarithmic functions, trigonometric and inverse trigonometric functions.

We end this section by stating an useful property of a continuous function.

Theorem 1.1.5 (Intermediate Value Theorem)**.** *Let f be a continuous function on $[a, b]$ and $f(a) \neq f(b)$. Then for any real number N between $f(a)$ and $f(b)$, there is a number c in (a, b) such that $f(c) = N$.*

1.2 Derivatives

Definition (Derivative)**.** The *derivative* of a function f at a, denoted by $f'(a)$ or $\left.\frac{df}{dx}\right|_{x=a}$, is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

when the limit exists. f is *differentiable* at a if $f'(a)$ is a real number. f is a *differentiable function* if f is differentiable at each point of its domain.

When we write $y = f(x)$, $f'(a)$ is also denoted by $y'(a)$ or $\left.\dfrac{dy}{dx}\right|_{x=a}$. An alternative limit form for $f'(a)$ is obtained by substituting $h = x - a$ where $h \to 0$ as $x \to a$.

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Example.

1. If $f(x) = x^2$ is differentiable at $x = 1$, find $f'(1)$.

   Solution. First we evaluate the following limit:

   $$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1^2}{x - 1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x\to 1}(x+1) = 2$$

   Since the above limit exists, $f(x) = x^2$ is differentiable at $x = 1$ and $f'(1) = 2$. The following is an alternative way of finding the above limit:

   $$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{(1+2h+h^2) - 1}{h} \\ &= \lim_{h\to 0} \frac{2h + h^2}{h} \\ &= \lim_{h\to 0}(2+h) \\ &= 2 \end{aligned}$$

   Note that $f(x) = x^2$ is a differentiable function because we can show that $f'(a) = 2a$ for any real number a.

2. Show that $f(x) = |x|$ is not differentiable at $x = 0$.

   Solution. First note that $f(x) = |x| = x$ for all $x \geq 0$ and $f(x) = |x| = -x$ for all $x < 0$.

   $$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x - 0}{x - 0} = \lim_{x\to 0^+} 1 = 1$$
   $$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{-x - 0}{x} = \lim_{x\to 0^-} (-1) = -1$$

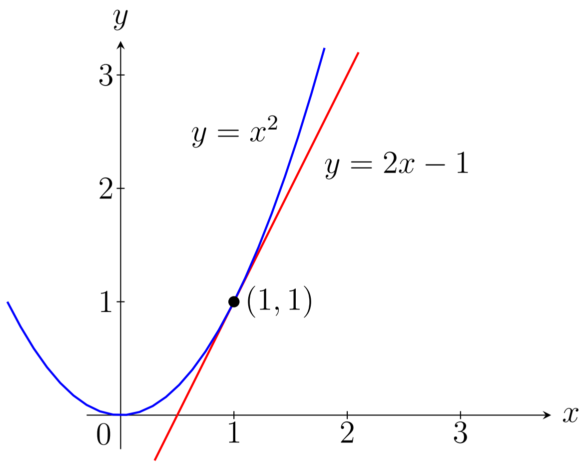
   Since the above left-hand limit and right-hand limit are not equal, the limit $\lim\limits_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0}$ does not exist and consequently $f(x) = |x|$ is not differentiable at $x = 0$.

   In general, a function f is not differentiable at a when there is a sharp corner in the graph of $y = f(x)$ at the point $(a, f(a))$.

Geometric interpretation of derivatives:
$f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. An equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a).$$

Example. Find an equation of the tangent line to the curve $y = x^2$ at $(1, 1)$.



Solution. For $f(x) = x^2$, $f'(x) = 2x$ and $f'(1) = 2$. Then an equation of the tangent line to $y = x^2$ at $(1, f(1)) = (1, 1)$ is

$$y - 1 = 2(x - 1) \implies y = 2x - 1.$$

The following theorem shows that differentiability implies continuity of a function.

Theorem 1.2.1. *If f is differentiable at a, then f is continuous at a.*

The following tables provide derivatives of some well-known functions:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^x	e^x
$a^x,\ a > 0$	$a^x(\ln a)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\ \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x\ \cot x$

$f(x)$	$f'(x)$
$\tan^{-1} x$	$\dfrac{1}{x^2+1}$
$\cot^{-1} x$	$-\dfrac{1}{x^2+1}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$-\dfrac{1}{x\sqrt{x^2-1}}$

The following rules help us differentiate combinations of multiple functions.

Differentiation rules

- $[cf(x)]' = cf'(x)$ (constant multiple rule)
- $(f+g)' = f' + g'$ (sum rule)
- $(f-g)' = f' - g'$ (difference rule)
- $(f \cdot g)' = f \cdot g' + f' \cdot g$ (product rule)
- $\left(\dfrac{f}{g}\right)' = \dfrac{g \cdot f' - f \cdot g'}{g^2}$ (quotient rule)
- If $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ (chain rule).
  Equivalently, if $F(x) = f[g(x)]$, then $F'(x) = f'[g(x)] \cdot g'(x)$

Example.

1. Find $f'(x)$ for $f(x) = 5x^2 \ln x - \dfrac{\sin x}{e^x}$.

   Solution.

$$\begin{aligned}
f'(x) &= \left(5x^2 \ln x - \frac{\sin x}{e^x}\right)' \\
&= (5x^2 \ln x)' - \left(\frac{\sin x}{e^x}\right)' && \text{(By the difference rule)} \\
&= 5(x^2 \ln x)' - \left(\frac{\sin x}{e^x}\right)' && \text{(By the constant multiple rule)} \\
&= 5\left(2x \cdot \ln x + x^2 \cdot \frac{1}{x}\right) - \left(\frac{e^x \cdot \cos x - \sin x \cdot e^x}{(e^x)^2}\right) && \text{(By the product and quotient rules)} \\
&= 10x \ln x + 5x - \left(\frac{e^x \cos x - e^x \sin x}{e^{2x}}\right) \\
&= 10x \ln x + 5x - e^{-x} \cos x + e^{-x} \sin x
\end{aligned}$$

2. Find $\dfrac{d}{dx}\left(e^{\sin x} + \cos(x^3)\right)$.

   Solution. By the sum and chain rules,

$$\begin{aligned}
\frac{d}{dx}\left(e^{\sin x} + \cos(x^3)\right) &= \frac{d}{dx}\left(e^{\sin x}\right) + \frac{d}{dx}\left(\cos(x^3)\right) \\
&= e^{\sin x} \cdot (\sin x)' - \sin(x^3) \cdot (x^3)' \\
&= e^{\sin x} \cdot \cos x - \sin(x^3) \cdot 3x^2 \\
&= e^{\sin x} \cos x - 3x^2 \sin(x^3).
\end{aligned}$$

Now we discuss some applications of derivatives starting with finding maximum value of a function over an interval.

Definition (Absolute maximum)**.** Let c be a number in the domain D of a function f. We call $f(c)$ the *absolute maximum* of f on D if $f(c) \geq f(x)$ for all x in D and the *absolute minimum* of f on D if $f(c) \leq f(x)$ for all x in D.

Example.

1. On $[0, 2\pi]$, $f(x) = \cos x$ has the absolute maximum $f(0) = f(2\pi) = 1$ and the absolute minimum $f(\pi) = -1$.

2. On $[-1, 2]$, $f(x) = -|x|$ has the absolute maximum $f(0) = 0$ and the absolute minimum $f(2) = -2$.

Definition (Critical number)**.** A *critical number* of a function f is a number in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist.

Example.

1. $c = \pi$ is a critical number of $f(x) = \cos x$ because $f'(\pi) = 0$.

2. $c = 0$ is a critical number of $f(x) = -|x|$ because $f'(0)$ does not exist.

Theorem 1.2.2. *Let f be a continuous function on $[a, b]$ with critical numbers $c_1, c_2, \ldots, c_k$. Then the absolute maximum of f on $[a, b]$ is the maximum of $f(a), f(b), f(c_1), f(c_2), \ldots, f(c_k)$. Similarly the absolute minimum of f on $[a, b]$ is the minimum of $f(a), f(b), f(c_1), f(c_2), \ldots, f(c_k)$.*

Example. With 12 meters of fencing, what is the largest area of a rectangular region that can be fenced in?

Solution. Suppose x and y are the length and width of the rectangular region in meters respectively. The perimeter is

$$2x + 2y = 12.$$

We would like to maximize the area xy. Substituting $y = 6 - x$, we get

$$xy = x(6 - x) = 6x - x^2.$$

So we find the absolute maximum of

$$f(x) = 6x - x^2$$

on $[0, 6]$. Note that x cannot be bigger than 6 because $x + y = 6$. Since f is a continuous function on $[0, 6]$, its absolute maximum could be attained at the critical numbers and the end points of $[0, 6]$.

$$f'(x) = 6 - 2x = 0 \implies x = 3 \text{ (critical number)}$$

We find the values of f at the end points of $[0, 6]$ and the critical number 3:

$$f(0) = 0, f(6) = 0, f(3) = 9$$

The absolute maximum of $f(x) = 6x - x^2$ on $[0, 6]$ is the maximum of $0, 9$ which is 9. When $x = 3$, $y = 3$. Thus the rectangular region with maximum area is a square with side 3 meters and area 9 square meters.

Derivatives are used to evaluate limits in certain forms.

Definition (Indeterminate form)**.** A limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an *indeterminate form of the type* $\frac{0}{0}$ if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Similarly, $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an *indeterminate form of the type* $\frac{\infty}{\infty}$ if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Here a can be $\pm\infty$ besides being a real number.

Example. $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$ is an indeterminate form of the type $\frac{0}{0}$ because

$$\lim_{x\to 0}(\cos x - 1) = \lim_{x\to 0} \sin x = 0.$$

$\lim_{x\to\infty} \frac{e^x}{x^2}$ is an indeterminate form of the type $\frac{\infty}{\infty}$ because

$$\lim_{x\to\infty} e^x = \lim_{x\to\infty} x^2 = \infty.$$

Theorem 1.2.3 (l'Hôpital's rule)**.** *Suppose* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is an indeterminate form of the type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$*. If f and g are differentiable on an open interval I containing a (except possibly at a) and $g'(x) \neq 0$ for all x in I, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit exists or equals $\pm\infty$.

Remark. l'Hôpital's rule is also valid when $x \to a^+$, $x \to a^-$, $x \to \infty$, and $x \to -\infty$.

Example. Evaluate (a) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$, (b) $\lim_{x\to\infty} \frac{x^2}{e^x}$.

Solution. (a) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$ is an indeterminate form of the type $\frac{0}{0}$. By l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{(\cos x - 1)'}{(\sin x)'} = \lim_{x\to 0} \frac{-\sin x}{\cos x} = \frac{\lim_{x\to 0}(-\sin x)}{\lim_{x\to 0} \cos x} = \frac{0}{1} = 0.$$

(b) $\lim_{x\to\infty} \frac{e^x}{x^2}$ is an indeterminate form of the type $\frac{\infty}{\infty}$. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x},$$

which is also an indeterminate form of the type $\frac{\infty}{\infty}$. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

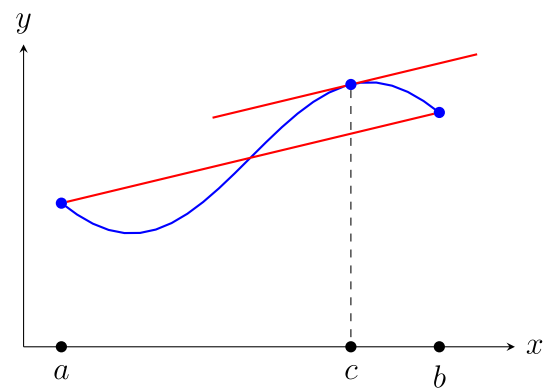
We end by mentioning two useful theorems regarding derivatives.

Theorem 1.2.4 (Rolle's Theorem)**.** *Let f be a function that is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$.*

Theorem 1.2.5 (Mean Value Theorem)**.** *Let f be a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there is a number c in (a,b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometric interpretation of the Mean Value Theorem:
If the curve $y = f(x)$ is continuous on $[a,b]$ and a tangent line to the curve can be drawn at $(x, f(x))$ for any x in (a,b), then the slope of the line joining the endpoints $(a, f(a))$ and $(b, f(b))$ is same as that of the tangent line at $(c, f(c))$ for some c in (a,b).



As an application of the Mean Value Theorem, the following theorem can be proved.

Theorem 1.2.6. *Let f be a function on an interval I.*

(a) If $f'(x) > 0$ on I, then f is increasing on I (i.e., $x_1 < x_2$ in $I \implies f(x_1) < f(x_2)$).

(b) If $f'(x) < 0$ on I, then f is decreasing on I (i.e., $x_1 < x_2$ in $I \implies f(x_1) > f(x_2)$).

(c) If $f''(x) > 0$ on I, then f is concave up on I (i.e., tangents are below the curve).

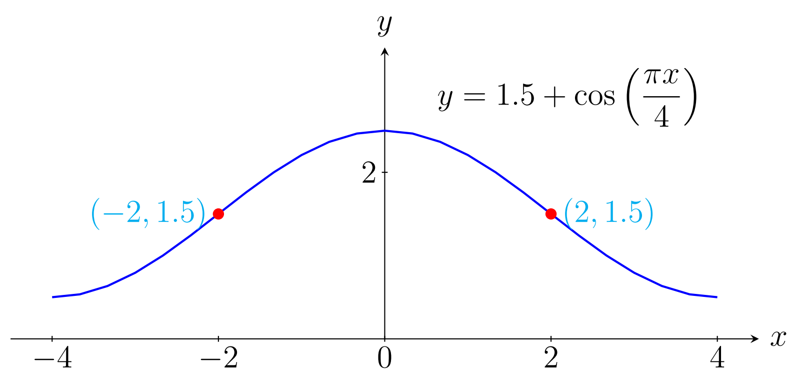
(d) If $f''(x) < 0$ on I, then f is concave down on I (i.e., tangents are above the curve).

Definition (Inflection point)**.** An *inflection point* of a curve $y = f(x)$ is a point $(c, f(c))$ at which the curve is continuous and its concavity changes.

Example. Consider $f(x) = 1.5 + \cos\left(\frac{\pi x}{4}\right)$ on $(-4, 4)$.

$$f'(x) = -\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right),\ f''(x) = -\left(\frac{\pi}{4}\right)^2 \cos\left(\frac{\pi x}{4}\right)$$

Note that $f'(x) = 0$ for $x = 0$ and $f''(x) = 0$ for $x = -2, 2$.
Since $f'(x) > 0$ on $(-4, 0)$, f is increasing on $(-4, 0)$. Since $f'(x) < 0$ on $(0, 4)$, f is decreasing on $(0, 4)$.
Since $f''(x) < 0$ on $(-2, 2)$, f is concave down on $(-2, 2)$. Since $f''(x) > 0$ on $(-4, -2) \cup (2, 4)$, f is concave up on $(-4, -2) \cup (2, 4)$. Also $(-2, 1.5)$ and $(2, 1.5)$ are inflection points of the curve $y = 1.5 + \cos\left(\frac{\pi x}{4}\right)$.



1.3 Integrals

Definition (Antiderivative)**.** An *antiderivative* of a function f on an interval I is a function F for which $F'(x) = f(x)$ for all $x \in I$. An antiderivative of f is also called an *indefinite integral* of f which is denoted by $\int f(x)\,dx$ and expressed as a general form of an antiderivative of f.

Example. Suppose $f(x) = 2x$ on $[-1, 1]$. Then $F(x) = x^2$ is an antiderivative of f because $F'(x) = 2x = f(x)$ for all x in $[-1, 1]$. Similarly $F(x) = x^2 + 3$ is an antiderivative of f. In fact, $F(x) = x^2 + C$ is an antiderivative of f for any constant C. We write the indefinite integral $\int f(x)\,dx$ as follows:

$$\int 2x\,dx = x^2 + C$$

We call C the *integration constant.* Also $f(x)$ in an indefinite integral $\int f(x)\,dx$ is called the *integrand.*

The following tables provide indefinite integrals of some well-known functions:

$f(x)$	$\int f(x)\,dx$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$a^x,\ a > 0, a \neq 1$	$\frac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$

$f(x)$	$\int f(x)\,dx$
$\sec^2 x$	$\tan x + C$
$\sec x\ \tan x$	$\sec x + C$
$\csc^2 x$	$-\cot x + C$
$\csc x\ \cot x$	$-\csc x + C$
$\frac{1}{x^2+1}$	$\tan^{-1} x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$

The following result helps us integrate combinations of multiple functions.

Theorem 1.3.1. *For integrable functions f and g and real number k, the following are true:* $\int (f \pm g)(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$ *and* $\int kf(x)\,dx = k\int f(x)\,dx$

Example.

$$\int (e^x - 2\sin x)\,dx = \int e^x\,dx - 2\int \sin x\,dx = e^x - 2(-\cos x) + C = e^x + 2\cos x + C$$

Note that we do not need to write an integration constant for each integration because multiple integration constants can be combined into one.

Definition (Definite integral)**.** The *definite integral* of a function $f : [a, b] \to \mathbb{R}$, denoted by $\int_a^b f(x)\,dx$, is a real number given by the following limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

when the limit exists for any choice of $x_i^* \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ where $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{n}$. When the definite integral $\int_a^b f(x)\,dx$ exists, f is called *integrable* over $[a, b]$.

Remark. a and b in $\int_a^b f(x)\,dx$ are called the *lower limit* and *upper limit* of integration respectively. We define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and $\int_a^a f(x)\,dx = 0$. The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called a *Riemann sum* which can be used to approximate $\int_b^a f(x)\,dx$.

Theorem 1.3.2. *If $f : [a,b] \to \mathbb{R}$ is continuous or has a finite number of removable or jump discontinuities, then f is integrable over $[a,b]$ and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right).$$

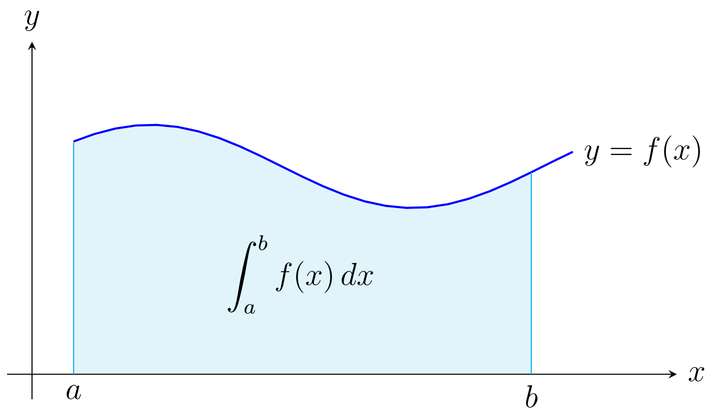
Example. Consider $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$. Since f is continuous over $[0,1]$, $\displaystyle\int_0^1 x^2\,dx$ exists and it is equal to

$$\begin{aligned}
\int_0^1 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1-0}{n} f\left(0 + i\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= \frac{1}{3}.
\end{aligned}$$

Remark. Compared to the above process of finding a definite integral using a limit, there is a better way of doing so using antiderivatives by the Fundamental Theorem of Calculus explained in the next section.

Geometric interpretation of definite integrals:

If $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ on $[a,b]$, more explicitly, the area of the region bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. Similarly if $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \leq 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx$ is the negative of the area above the curve $y = f(x)$ and below the x-axis on $[a,b]$. In general, $\displaystyle\int_a^b f(x)\,dx$ is the net signed area for $y = f(x)$ over $[a,b]$, i.e., the area under minus above the curve $y = f(x)$.



Example. $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$ is the area of the region bounded by the parabola $y = x^2$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Theorem 1.3.3. *Let f and g be integrable functions over $[a,b]$ and k be a real number. Then the following are true.*

(a) $\displaystyle\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ *and* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$

(b) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all c in $[a,b]$.*

Example.

1. Evaluate $\displaystyle\int_0^1 (x^2 - 3x)\,dx.$

   Solution. By the above properties,
$$\int_0^1 (x^2 - 3x)\,dx = \int_0^1 x^2\,dx + \int_0^1 (-3x)\,dx = \int_0^1 x^2\,dx - 3\int_0^1 x\,dx.$$
   We found $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$ by the limit definition of a definite integral before. Note that $\displaystyle\int_0^1 x\,dx = \frac{1}{2}$ is the area of the triangular region below the line $y = x$ and above the x-axis between the vertical lines $x = 0$ and $x = 1$. Thus
$$\int_0^1 (x^2 - 3x)\,dx = \int_0^1 x^2\,dx - 3\int_0^1 x\,dx = \frac{1}{3} - 3\frac{1}{2} = -\frac{7}{6}.$$

2. Evaluate $\displaystyle\int_0^3 (1 - x)\,dx$ as an area.

   Solution. The line $y = 1 - x$ intersects the x-axis at $x = 1$. Then
$$\int_0^3 (1-x)\,dx = \int_0^1 (1-x)\,dx + \int_1^3 (1-x)\,dx = \frac{1}{2}\cdot 1\cdot 1 - \frac{1}{2}\cdot 2\cdot 2 = -\frac{3}{2}.$$

Note that $\int_1^3 (1-x)\,dx = -2$ is the negative of the area of the triangular region above the line $y = 1-x$ and below the x-axis between the vertical lines $x = 0$ and $x = 1$.

1.4 Fundamental Theorem of Calculus

In this section we learn the Fundamental Theorem of Calculus (abbreviated by FTC) which shows that derivative and integration are reverse processes.

The first part of the Fundamental Theorem of Calculus (abbreviated by FTC-I) finds an antiderivative or an indefinite integral of a continuous function.

Theorem 1.4.1 (FTC-I). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Consider the function $F : [a,b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt.$$

Then for all x in (a,b), $F'(x) = f(x)$, i.e.,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example. For all x in $(0,1)$, evaluate

$$\frac{d}{dx}\int_0^x e^{t^2}\,dt.$$

Solution. Here $f(t) = e^{t^2}$ which is continuous on $[0,1]$. Then by the FTC-I,

$$\frac{d}{dx}\int_0^x e^{t^2}\,dt = f(x) = e^{x^2}.$$

The second part of the Fundamental Theorem of Calculus (abbreviated by FTC-II) finds a definite integral of a continuous function.

Theorem 1.4.2 (FTC-II). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function with an antiderivative F on $[a,b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

which is denoted by $F(x)|_a^b$.

Example. Evaluate $\displaystyle\int_0^{\pi} \sin x\,dx$ and $\displaystyle\int_0^{2\pi} \sin x\,dx$.

Solution. An antiderivative of $f(x) = \sin x$ is $F(x) = -\cos x$. Then by the FTC-II,

$$\int_0^{\pi} \sin x\,dx = F(x)\big|_0^{\pi} = F(\pi) - F(0) = -\cos(\pi) + \cos(0) = 2.$$

Similarly

$$\int_0^{2\pi} \sin x\,dx = -\cos x\big|_0^{2\pi} = -\cos(2\pi) + \cos(0) = 0$$

which is evident from interpreting the integral as a net signed area.

Chapter 2

Techniques of Integration

In this chapter we learn various techniques of integration.

2.1 Integration by Substitution

Integration by substitution is a rule of integration which is also known as u-substitution or change of variables.

The substitution rule/ u-substitution: If f is a continuous function on an interval I and $u = g(x)$ is a differentiable function whose range is inside I where g' is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Note that $g'(x)\,dx = du$ because $\dfrac{du}{dx} = g'(x)$. For further justifications, suppose F is an antiderivative of f, i.e., $F' = f$. By the chain rule,

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x).$$

Then

$$\int f(g(x))g'(x)\,dx = \int (F(g(x)))'\,dx = F(g(x)) + C.$$

Substituting $u = g(x)$ and $f = F'$, we get

$$\int f(g(x))g'(x)\,dx = F(u) + C = \int F'(u)\,du = \int f(u)\,du.$$

Example. Evaluate $\displaystyle\int xe^{x^2}\,dx$.

Solution. Let $u = x^2$. Then $du = 2xdx$ and $xdx = \frac{1}{2}du$.

$$\begin{aligned}
\int xe^{x^2}\,dx &= \int e^{x^2} x\,dx \\
&= \int e^u \frac{1}{2} du && \text{(By the u-substitution)} \\
&= \frac{1}{2}\int e^u\,du \\
&= \frac{e^u}{2} + C \\
&= \frac{e^{x^2}}{2} + C
\end{aligned}$$

Example. Evaluate $\int \tan x\,dx$.

Solution.

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\
&= \int \frac{1}{u}(-du) && \text{(Let $u = \cos x$. Then $du = -\sin x\,dx \implies \sin x\,dx = -du$)} \\
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= \ln\left(\frac{1}{|u|}\right) + C \\
&= \ln\left(\frac{1}{|\cos x|}\right) + C \\
&= \ln(|\sec x|) + C
\end{aligned}$$

Example. Evaluate $\int_0^{\sqrt{5}} \frac{2x}{\sqrt{x^2+4}}\,dx$.

Solution. First we do the corresponding indefinite integral:

$$\begin{aligned}
\int \frac{2x}{\sqrt{x^2+4}}\,dx &= \int \frac{du}{\sqrt{u}} && \text{(Let $u = x^2+4$. Then $du = 2x\,dx$)} \\
&= 2\sqrt{u} + C \\
&= 2\sqrt{x^2+4} + C
\end{aligned}$$

Then

$$\int_0^{\sqrt{5}} \frac{2x}{\sqrt{x^2+4}}\,dx = 2\sqrt{x^2+4}\Big|_0^{\sqrt{5}} = 2\sqrt{(\sqrt{5})^2+4} - 2\sqrt{4} = 6 - 4 = 2.$$

The preceding definite integral can also be evaluated by the following **substitution rule/ u-substitution for definite integrals:**

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note that when $x = a$, $u = g(a)$ and when $x = b$, $u = g(b)$.

Example. Evaluate $\displaystyle\int_0^{\sqrt{5}} \frac{2x}{\sqrt{x^2+4}}\,dx$.

Solution. We substitute $u = x^2 + 4$ as before. Then $du = 2x\,dx$. When $x = 0$, $u = 4$ and when $x = \sqrt{5}$, $u = 9$.

$$\begin{aligned}\int_0^{\sqrt{5}} \frac{2x}{\sqrt{x^2+4}}\,dx &= \int_4^9 \frac{du}{\sqrt{u}} \\ &= 2\sqrt{u}\Big|_4^9 \\ &= 2\sqrt{9} - 2\sqrt{4} \\ &= 2\end{aligned}$$

Exercises

1. Evaluate the following integrals:

   (a) $\displaystyle\int 6x\sin(x^2)\,dx$

   (b) $\displaystyle\int_1^e \frac{\ln x}{3x}\,dx$

   (c) $\displaystyle\int \frac{\sin(2x)}{1+\sin^2 x}\,dx$

   (d) $\displaystyle\int \frac{2\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

   (e) $\displaystyle\int x^3\sqrt{x^2+5}\,dx$

   (f) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \sin(\cos x)\,dx$

2. Prove that $\displaystyle\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$ for $a \neq 0$. This implies many formulas such as

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,\ \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1,$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C,\ \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C,$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C,\ \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Answers

1. (a) $-3\cos(x^2) + C$
   (b) $\dfrac{1}{6}$
   (c) $\ln(1 + \sin^2 x) + C$
   (d) $(\sin^{-1} x)^2 + C$
   (e) $\dfrac{1}{5}(x^2+5)^{\frac{5}{2}} - \dfrac{5}{3}(x^2+5)^{\frac{3}{2}} + C$
   (f) $1 - \cos 1$

2. Substitute $u = ax + b$

2.2 Integration by Parts

Integration by parts: If f and g are differentiable functions such that f' and g' are integrable, then

$$\int fg'\,dx = fg - \int f'g\,dx.$$

Equivalently,

$$\int u\,dv = uv - \int v\,du.$$

Proof. By the product rule, $(fg)' = f'g + fg'$. Integrating both sides we get,

$$\int (fg)'\,dx = \int (f'g + fg')\,dx \implies fg = \int f'g\,dx + \int fg'\,dx.$$

Subtracting $\int f'g\,dx$ from both sides, we get

$$\int fg'\,dx = fg - \int f'g\,dx.$$

The rest follows by the substitution $u = f$ and $dv = g'\,dx$ for which $du = f'\,dx$ and $v = \int dv = \int g'\,dx = g$. □

Example. Evaluate $\displaystyle\int xe^x\,dx$.

Solution.

$$\begin{aligned}
\int xe^x\,dx &= \int u\,dv && (\text{Let } u = x,\ dv = e^x\,dx)\\
&= uv - \int v\,du && \left(du = dx,\ v = \int e^x\,dx = e^x\right)\\
&= xe^x - \int e^x\,dx\\
&= xe^x - e^x + C
\end{aligned}$$

Note that the choice of u and v is important. Integration by parts does not produce any simple answer if we chose $u = e^x$ and $dv = x\,dx$. Usually we choose u so that its derivative is "simpler" than u and dv is integrable. Some like the following decreasing order of preference for u: LIATE (Logarithm, Inverse Trig, Algebraic (Polynomial), Trig, Exponential).

Example. Evaluate $\displaystyle\int \ln x\,dx$.

Solution.

$$\begin{aligned}
\int \ln x\,dx &= \int u\,dv && (\text{Let } u = \ln x,\ dv = dx)\\
&= uv - \int v\,du && \left(du = \frac{1}{x}\,dx,\ v = \int dx = x\right)\\
&= (\ln x)x - \int x\frac{1}{x}\,dx\\
&= (\ln x)x - \int dx\\
&= x\ln x - x + C
\end{aligned}$$

Circular integration by parts: Sometimes we need to do integration by parts more than once and get the same integral back.

Example. Evaluate $\displaystyle\int e^x \sin x\,dx$.

Solution.

$$\begin{aligned}
\int e^x \sin x\,dx &= \int u\,dv && (\text{Let } u = \sin x,\ dv = e^x\,dx)\\
&= uv - \int v\,du && \left(du = \cos x\,dx,\ v = \int e^x\,dx = e^x\right)\\
&= \sin x\, e^x - \int e^x \cos x\,dx. && (1)
\end{aligned}$$

Now we apply integration by parts on $\int e^x \cos x\,dx$. Let $u = \cos x$, $dv = e^x\,dx$. Then $du = -\sin x\,dx$, $v = \int e^x\,dx = e^x$.

$$\begin{aligned}
\int e^x \cos x\,dx &= \int u\,dv\\
&= uv - \int v\,du\\
&= \cos x\, e^x - \int e^x(-\sin x)\,dx\\
&= \cos x\, e^x + \int e^x \sin x\,dx.
\end{aligned}$$

Plugging this in (1), we get

$$\begin{aligned}\int e^x \sin x\, dx &= \sin x\, e^x - \int e^x \cos x\, dx \\ &= \sin x\, e^x - \left(\cos x e^x + \int e^x \sin x\, dx\right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x\, dx.\end{aligned}$$

Adding $\int e^x \sin x\, dx$ to both sides, we get

$$\begin{array}{rrcl} & 2\int e^x \sin x\, dx &=& e^x(\sin x - \cos x) + C \\ \implies & \int e^x \sin x\, dx &=& \frac{1}{2}e^x(\sin x - \cos x) + C.\end{array}$$

Reduction Formulas: Integrating by parts in a circular fashion we get the following reduction formulas:

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx,\ n \geq 2.$$

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx,\ n \geq 2.$$

Proof.

$$\begin{aligned}\int \cos^n x\, dx = \int u\, dv\ &\big(\text{Let } u = \cos^{n-1} x,\ dv = \cos x\, dx\big) \\ &= uv - \int v\, du\ \left(du = (n-1)\cos^{n-2} x(-\sin x)\, dx,\ v = \int \cos x\, dx = \sin x\right) \\ &= \cos^{n-1} x \sin x - \int \sin x(n-1)\cos^{n-2} x(-\sin x)\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (\cos^{n-2} x - \cos^n x)\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx\end{aligned}$$

Adding $(n-1)\int \cos^n x\, dx$ to both sides, we get

$$\begin{array}{rrcl} & n\int \cos^n x\, dx &=& \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx \\ \implies & \int \cos^n x\, dx &=& \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.\end{array}$$

The other formula has a similar proof. □

Example. Evaluate $\displaystyle\int \cos^4 x\,dx$.

Solution.

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx$$

We can also apply the reduction formula to $\displaystyle\int \cos^2 x\,dx$. Alternatively we can use the trig identity $\cos^2 x = \dfrac{1+\cos(2x)}{2}$. Then

$$\begin{aligned}\int \cos^4 x\,dx &= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx \\ &= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\int (1+\cos(2x))\,dx \\ &= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\left(x + \frac{\sin(2x)}{2}\right) + C\end{aligned}$$

Exercises

1. Evaluate the following integrals:

   (a) $\displaystyle\int x^2 e^{4x}\ dx$

   (b) $\displaystyle\int x^2 \ln x\ dx$

   (c) $\displaystyle\int x(\ln x)^2\ dx$

   (d) $\displaystyle\int_0^1 \tan^{-1} x\ dx.$

   (e) $\displaystyle\int 2x\tan^{-1} x\ dx$

   (f) $\displaystyle\int \cos^{-1} x\ dx$

   (g) $\displaystyle\int x^2\cos(2x)\ dx$

   (h) $\displaystyle\int e^x \cos x\ dx$

   (i) $\displaystyle\int \sin^3 x\ dx$

   (j) $\displaystyle\int \sec^3 x\ dx$

   (k) $\displaystyle\int_0^1 x3^x\ dx$

2. Prove that

$$\int \sec^n x\,dx = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \geq 2.$$

Answers

1. (a) $\frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$ (b) $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$ (c) $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$ (d) $\frac{\pi}{4} - \frac{\ln 2}{2}$ (e) $x^2\tan^{-1} x + \tan^{-1} x - x + C$ (f) $x\cos^{-1} x - \sqrt{1-x^2} +$

C (g) $\frac{1}{2}x^2\sin(2x)+\frac{1}{2}x\cos(2x)-\frac{1}{4}\sin(2x)+C$ (h) $\frac{1}{2}e^x(\cos x+\sin x)+C$ (i) $-\frac{1}{3}\sin^2 x\cos x-\frac{2}{3}\cos x+C$ or $\frac{1}{3}\cos^3 x-\cos x+C$ (j) $\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+C$ (k) $\frac{3}{\ln 3}-\frac{2}{(\ln 3)^2}$

2. Apply circular by parts

2.3 Trigonometric Integrals

In this section we learn to evaluate trigonometric integrals of the form:

$$\int \sin^m x\cos^n x\,dx,\ m,n\geq 0$$

$$\int \tan^m x\sec^n x\,dx,\ m,n\geq 0$$

There are three cases for $\int \sin^m x\cos^n x\,dx$:

1. The power of sine is odd, i.e., $m=2k+1$ for some integer $k\geq 0$.

   Steps:

$$\begin{aligned}\int \sin^{2k+1} x\cos^n x\,dx &= \int (\sin^2 x)^k\cos^n x\sin x\,dx\\ &= \int (1-\cos^2 x)^k\cos^n x\sin x\,dx\\ &= \int (1-u^2)^k u^n\,(-du)\ \text{(Let } u=\cos x\text{. Then } du=-\sin x\,dx)\end{aligned}$$

   Example. Evaluate $\int \sin^3 x\cos x\,dx$.

Solution.

$$\begin{aligned}
\int \sin^3 x \cos x \, dx &= \int \sin^2 x \cos x \sin x \, dx \\
&= \int (1-\cos^2 x) \cos x \sin x \, dx \\
&= \int (1-u^2) u \, (-du) \ \ (\text{Let } u = \cos x. \text{ Then } du = -\sin x \, dx) \\
&= \int (-u + u^3) \, du \\
&= -\frac{u^2}{2} + \frac{u^4}{4} + C \\
&= -\frac{1}{2}\cos^2 x + \frac{1}{4}\cos^4 x + C.
\end{aligned}$$

2. The power of cosine is odd, i.e., $n = 2k+1$ for some integer $k \geq 0$.

   Steps:

$$\begin{aligned}
\int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x (\cos^2 x)^k \cos x \, dx \\
&= \int \sin^m x (1-\sin^2 x)^k \cos x \, dx \\
&= \int u^m (1-u^2)^k \, du \ \ (\text{Let } u = \sin x. \text{ Then } du = \cos x \, dx)
\end{aligned}$$

Example. Evaluate $\int \sin^2 x \cos^5 x \, dx$.

Solution.

$$\begin{aligned}
\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x (\cos^2 x)^2 \cos x \, dx \\
&= \int \sin^2 x (1-\sin^2 x)^2 \cos x \, dx \\
&= \int u^2 (1-u^2)^2 \, du \ \ (\text{Let } u = \sin x. \text{ Then } du = \cos x \, dx) \\
&= \int u^2 (1 - 2u^2 + u^4) \, du \\
&= \int (u^2 - 2u^4 + u^6) \, du \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.
\end{aligned}$$

3. The powers of sine and cosine are both even.

   Steps: Write the integrand as an expression of cosines using the following trigonometric identities.

$$\begin{aligned}\cos^2 x &= \frac{1}{2}(1+\cos(2x))\\ \sin^2 x &= \frac{1}{2}(1-\cos(2x))\\ \cos x \cos y &= \frac{1}{2}[\cos(x+y)+\cos(x-y)]\end{aligned}$$

   Example. Evaluate $\int \sin^2 x \cos^2 x \, dx$.

   Solution.

$$\begin{aligned}\sin^2 x \cos^2 x &= \frac{1}{2}(1-\cos(2x))\,\frac{1}{2}(1+\cos(2x))\\ &= \frac{1}{4}\left(1-\cos^2(2x)\right)\\ &= \frac{1}{4}\left[1-\frac{1}{2}(1+\cos(4x))\right]\\ &= \frac{1}{8}[1-\cos(4x)].\end{aligned}$$

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{8}[1-\cos(4x)]\ dx\\ &= \frac{1}{8}\int [1-\cos(4x)]\ dx\\ &= \frac{1}{8}\left[x-\frac{\sin(4x)}{4}\right]+C\\ &= \frac{x}{8}-\frac{1}{32}\sin(4x)+C.\end{aligned}$$

   For an alternative solution, we use the identity $\sin(2x) = 2\sin x \cos x$ which implies

$$\sin^2 x \cos^2 x = (\sin x \cos x)^2 = \frac{\sin^2(2x)}{4} = \frac{1}{8}[1-\cos(4x)].$$

Note that trigonometric (product-to-sum) identities can also be used to evaluate integrals of the form:

$$\int \sin(mx)\cos(nx)\ dx,\ \int \sin(mx)\sin(nx)\ dx,\ \int \cos(mx)\cos(nx)\ dx.$$

Example. Evaluate $\int \cos(3x)\cos(5x)\,dx$.

Solution.

$$\begin{aligned}\cos(3x)\cos(5x) &= \frac{1}{2}\left[\cos(3x+5x)+\cos(3x-5x)\right]\\ &= \frac{1}{2}\left[\cos(8x)+\cos(-2x)\right]\\ &= \frac{1}{2}\left[\cos(8x)+\cos(2x)\right].\end{aligned}$$

$$\begin{aligned}\int \cos(3x)\cos(5x)\,dx &= \int \frac{1}{2}\left[\cos(8x)+\cos(2x)\right]\,dx\\ &= \frac{1}{2}\int \left[\cos(8x)+\cos(2x)\right]\,dx\\ &= \frac{1}{2}\left[\frac{\sin(8x)}{8}+\frac{\sin(2x)}{2}\right]+C\\ &= \frac{1}{16}\sin(8x)+\frac{1}{4}\sin(2x)+C.\end{aligned}$$

To evaluate $\int \tan^m x \sec^n x\,dx$, first recall the following formulas:

$$\begin{aligned}\int \tan x\,dx &= \ln|\sec x|+C\\ \int \sec x\,dx &= \ln|\sec x+\tan x|+C\end{aligned}$$

1. $m = 2k+1$ for some integer $k \geq 0$ and $n \geq 1$.

   Steps:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x\,dx &= \int (\tan^2 x)^k \tan x \sec^{n-1} x \sec x\,dx\\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx\\ &= \int (u^2-1)^k u^{n-1}\,du \quad (\text{Let } u = \sec x.\text{ Then } du = \sec x \tan x\,dx)\end{aligned}$$

   Example. Evaluate $\int \tan^3 x \sec^2 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^3 x \sec^2 x \, dx &= \int \tan^2 x \tan x \sec x \sec x \, dx \\ &= \int (\sec^2 x - 1) \sec x \sec x \tan x \, dx \\ &= \int (u^2 - 1) u \, du \quad (\text{Let } u = \sec x. \text{ Then } du = \sec x \tan x \, dx) \\ &= \int (u^3 - u) \, du \\ &= \frac{u^4}{4} - \frac{u^2}{2} + C \\ &= \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C.\end{aligned}$$

2. $n = 2k$ for some integer $k \geq 1$.

Steps:

$$\begin{aligned}\int \tan^m x \sec^{2k} x \, dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx \\ &= \int u^m (1 + u^2)^{k-1} \, du \quad \big(\text{Let } u = \tan x. \text{ Then } du = \sec^2 x \, dx\big)\end{aligned}$$

Example. Evaluate $\displaystyle\int \tan^2 x \sec^4 x \, dx$.

Solution.

$$\begin{aligned}\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\ &= \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx \\ &= \int u^2 (1 + u^2) \, du \quad \big(\text{Let } u = \tan x. \text{ Then } du = \sec^2 x \, dx\big) \\ &= \int (u^2 + u^4) \, du \\ &= \frac{u^3}{3} + \frac{u^5}{5} + C \\ &= \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C.\end{aligned}$$

3. m is even and n is odd.

Steps: Write the integrand as an expression of powers of secants by using the identity $\tan^2 x = \sec^2 x - 1$ and then apply the following reduction formula:

$$\int \sec^n x\,dx = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \geq 2.$$

Example. Evaluate $\displaystyle\int \tan^2 x \sec^3 x\,dx$.

Solution.

$$\begin{aligned}
\int \tan^2 x \sec^3 x\,dx &= \int (\sec^2 x - 1)\sec^3 x\,dx \\
&= \int (\sec^5 x - \sec^3 x)\,dx \\
&= \int \sec^5 x\,dx - \int \sec^3 x\,dx \\
&= \frac{1}{4}\tan x \sec^3 x + \frac{3}{4}\int \sec^3 x\,dx - \int \sec^3 x\,dx \\
&= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\int \sec^3 x\,dx \\
&= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\left(\frac{1}{2}\tan x \sec x + \frac{1}{2}\int \sec x\,dx\right) \\
&= \frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\tan x \sec x - \frac{1}{8}\ln|\sec x + \tan x| + C.
\end{aligned}$$

Exercises

Evaluate the following trigonometric integrals:

1. $\displaystyle\int \sin^2 x \cos^3 x\,dx$
2. $\displaystyle\int \sin^5 x\,dx$
3. $\displaystyle\int \cos^4 x\,dx$
4. $\displaystyle\int \sin(3x)\sin(4x)\,dx$
5. $\displaystyle\int \sec^4 x\,dx.$
6. $\displaystyle\int \tan^6 x \sec^4 x\,dx$
7. $\displaystyle\int \sin^5 x\,\sqrt[3]{\cos x}\,dx$
8. $\displaystyle\int \frac{\sin^4 x}{\cos^2 x}\,dx$
9. $\displaystyle\int \frac{\tan^5 x}{\sqrt{\sec x}}\,dx$
10. $\displaystyle\int \cot^3 x\,dx$
11. $\displaystyle\int \csc^4 x\,\cot^6 x\,dx$
12. $\displaystyle\int \sin(6x)\,\cos(4x)\,dx$
13. $\displaystyle\int \cos(7x)\,\cos(4x)\,dx$
14. $\displaystyle\int 2x\sin^2 x\,dx$

Answers

1. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$
2. $-\left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right) + C$
3. $\frac{1}{4}\left[\sin(2x) + \frac{3x}{2} + \frac{\sin(4x)}{8}\right] + C$
4. $\frac{1}{2}\sin x - \frac{1}{14}\sin(7x) + C$
5. $\frac{1}{3}\tan^3 x + \tan x + C$
6. $\frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$
7. $-\frac{3}{4}(\cos x)^{\frac{4}{3}} + \frac{3}{5}(\cos x)^{\frac{10}{3}} - \frac{3}{16}(\cos x)^{\frac{16}{3}} + C$
8. $\tan x - \frac{3}{2}x + \frac{1}{4}\sin(2x) + C$
9. $\frac{2}{7}(\sec x)^{\frac{7}{2}} - \frac{4}{3}(\sec x)^{\frac{3}{2}} - \frac{2}{\sqrt{\sec x}} + C$
10. $-\frac{1}{2}\csc^2 x - \ln|\sin x| + C$
11. $-\frac{1}{7}\cot^7 x - \frac{1}{9}\cot^9 x + C$
12. $-\frac{1}{20}\cos(10x) - \frac{1}{4}\cos(2x) + C$
13. $\frac{1}{22}\sin(11x) + \frac{1}{6}\sin(3x) + C$
14. $\frac{1}{2}x^2 - \frac{1}{2}x\sin(2x) - \frac{1}{4}\cos(2x) + C$

2.4 Trigonometric Substitution

In this section we learn to evaluate integrals involving one of the following expressions:

$$\sqrt{x^2+a^2},\ \sqrt{x^2-a^2},\ \sqrt{a^2-x^2}$$

We use the following trigonometric substitutions:

$\sqrt{a^2-x^2} = a\cos\theta$	for $x = a\sin\theta$	where $dx = a\cos\theta\ d\theta$
$\sqrt{x^2-a^2} = a\tan\theta$	for $x = a\sec\theta$	where $dx = a\sec\theta\tan\theta\ d\theta$
$\sqrt{x^2+a^2} = a\sec\theta$	for $x = a\tan\theta$	where $dx = a\sec^2\theta\ d\theta$

Example. Show that $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C,\ a > 0.$

Solution. Let $x = a\sin\theta$. Then $dx = a\cos\theta\ d\theta$ and

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} = \frac{1}{\sqrt{a^2(1-\sin^2\theta)}} = \frac{1}{\sqrt{a^2\cos^2\theta}} = \frac{1}{a\cos\theta}.$$

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2-x^2}} &= \int \frac{1}{a\cos\theta}\, a\cos\theta\, d\theta \\ &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1}\left(\frac{x}{a}\right) + C.\end{aligned}$$

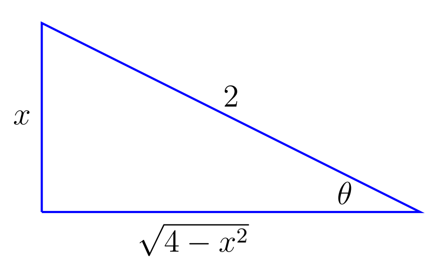
Example. Evaluate $\int \sqrt{4-x^2}\,dx$.

Solution. Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta.$$

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= \int 2\cos\theta\; 2\cos\theta\, d\theta \\ &= 2\int 2\cos^2\theta\, d\theta \\ &= 2\int (1+\cos(2\theta))\, d\theta \\ &= 2\left(\theta + \frac{\sin(2\theta)}{2}\right) + C \\ &= 2\theta + 2\sin\theta\cos\theta + C\end{aligned}$$

Now we convert the answer to an expression in terms of x. Since $\sin\theta = \frac{x}{2}$, $\theta = \sin^{-1}\left(\frac{x}{2}\right)$. To get $\cos\theta$, draw a right triangle with an angle θ where $\sin\theta = \frac{x}{2}$. Since $\sin\theta$ is the ratio of the lengths of the opposite side and the hypotenuse, we label the opposite side and the hypotenuse with x and 2 respectively. Then by the Pythagorean Theorem, the length of the adjacent side is $\sqrt{4-x^2}$. Consequently $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.



$\sin\theta = \frac{x}{2}$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

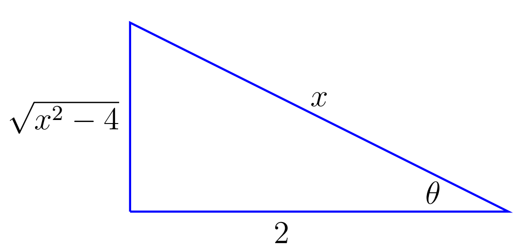
$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= 2\theta + 2\sin\theta\cos\theta + C\\ &= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\frac{x}{2}\frac{\sqrt{4-x^2}}{2} + C\\ &= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C.\end{aligned}$$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$.

Solution. Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\; d\theta$ and

$$\frac{1}{\sqrt{x^2-4}} = \frac{1}{\sqrt{4\sec^2\theta - 4}} = \frac{1}{\sqrt{4\tan^2\theta}} = \frac{1}{2\tan\theta}.$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-4}} &= \int \frac{1}{2\tan\theta} 2\sec\theta\tan\theta\; d\theta\\ &= \int \sec\theta\; d\theta\\ &= \ln|\sec\theta + \tan\theta| + C'\\ &= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C'\\ &= \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C'\\ &= \ln\left|x+\sqrt{x^2-4}\right| - \ln 2 + C'\\ &= \ln\left|x+\sqrt{x^2-4}\right| + C\;\;(\text{where } C = C' - \ln 2)\,.\end{aligned}$$



$\sec\theta = \frac{x}{2}$

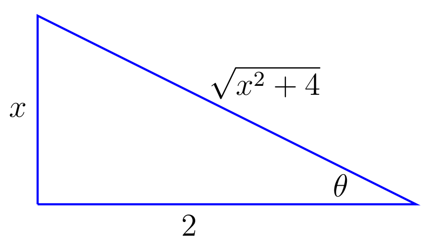
$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

Example. Evaluate $\int \sqrt{x^2+4}\,dx$.

Solution. Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\; d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

$$\begin{aligned}
\int \sqrt{x^2+4}\,dx &= \int 2\sec\theta\, 2\sec^2\theta\, d\theta \\
&= 4\int \sec^3\theta\, d\theta \\
&= 4\cdot\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C' \\
&= 2\left(\frac{\sqrt{x^2+4}}{2}\frac{x}{2} + \ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|\right) + C' \\
&= \frac{x\sqrt{x^2+4}}{2} + 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C' \\
&= \frac{x\sqrt{x^2+4}}{2} + 2\ln\left|x+\sqrt{x^2+4}\right| - 2\ln 2 + C' \\
&= \frac{x\sqrt{x^2+4}}{2} + 2\ln\left|x+\sqrt{x^2+4}\right| + C \ \ (\text{where } C = C' - 2\ln 2)\,.
\end{aligned}$$



$\tan\theta = \frac{x}{2}$

$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

Note that if the integrand involves $\sqrt{ax^2+bx+c}$, then we complete the square under the square root and apply the previous techniques of trigonometric substitutions. Recall:

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+13}}$.

Solution. First we complete the square: $x^2-6x+13 = (x-3)^2+2^2$. Let $x-3 = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2-6x+13} = \sqrt{(x-3)^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x^2-6x+13}} &= \int \frac{2\sec^2\theta\; d\theta}{2\sec\theta} \\
&= \int \sec\theta\; d\theta \\
&= \ln|\sec\theta+\tan\theta| + C' \\
&= \ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2+4}}{2}\right| + C' \\
&= \ln\left|\frac{x-3+\sqrt{x^2-6x+13}}{2}\right| + C' \\
&= \ln\left|x-3+\sqrt{x^2-6x+13}\right| - \ln 2 + C' \\
&= \ln\left|x-3+\sqrt{x^2-6x+13}\right| + C \;\; (\text{where } C = C' - \ln 2)\,.
\end{aligned}
$$

Exercises

Evaluate the following integrals by trigonometric substitution:

1. $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$
2. $\displaystyle\int \sqrt{1-4x^2}\; dx$
3. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\; dx$
4. $\displaystyle\int \sqrt{5+4x-x^2}\; dx$
5. $\displaystyle\int \frac{\sqrt{9-16x^2}}{x^2}\; dx$
6. $\displaystyle\int \frac{x}{\sqrt{x^2+2x}}\; dx$
7. $\displaystyle\int \frac{x^3}{\sqrt{9x^2+4}}\; dx$
8. $\displaystyle\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\; dx$

Answers

1. $\ln|\sqrt{x^2+16}+x| + C$
2. $\dfrac{\sin^{-1}(2x)}{4} + \dfrac{x\sqrt{1-4x^2}}{2} + C$
3. $\dfrac{\sec^{-1}(x/3)}{6} - \dfrac{\sqrt{x^2-9}}{2x^2} + C$
4. $\dfrac{9\sin^{-1}(\frac{x-2}{3})}{2} + \dfrac{(x-2)\sqrt{5+4x-x^2}}{2} + C$
5. $-\dfrac{\sqrt{9-16x^2}}{x} - 4\sin^{-1}\left(\dfrac{4x}{3}\right) + C$
6. $\sqrt{x^2+2x} - \ln|x+1+\sqrt{x^2+2x}| + C$
7. $\dfrac{1}{243}\sqrt{(9x^2+4)^3} - \dfrac{4}{81}\sqrt{9x^2+4} + C$
8. $\dfrac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

2.5 Partial Fractions

In this section we learn to integrate rational functions:

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where P and Q are polynomials. The basic steps are as follows:

1. If $\deg P \geq \deg Q$, use polynomial long division to write

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

   where $\deg R < \deg Q$.

2. Write $\dfrac{R(x)}{Q(x)}$ as a sum of partial fractions of the form $\dfrac{A}{(ax+b)^n}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^n}$.

The preceding step 2 will be elaborated with examples in four cases:

1. $Q(x)$ is a product of distinct linear factors, i.e.,

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)}.$$

   In this case, the partial fraction decomposition takes the following form:

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

   Example. Evaluate $\displaystyle\int \frac{2x^3+x^2-4x+7}{x^2+x-2}\,dx$.

   Solution. By polynomial long division, we get

$$\frac{2x^3+x^2-4x+7}{x^2+x-2} = 2x-1+\frac{x+5}{x^2+x-2}.$$

   We factor the denominator: $x^2+x-2 = (x-1)(x+2)$. For a partial fraction decomposition, let

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

   Multiplying by $(x-1)(x+2)$, we get

$$x+5 = A(x+2) + B(x-1) \qquad (1)$$

Plugging $x = 1$ in (1), we get

$$1 + 5 = A(1+2) + B(1-1) \implies 6 = 3A \implies A = \frac{6}{3} = 2.$$

Plugging $x = -2$ in (1), we get

$$-2 + 5 = A(-2+2) + B(-2-1) \implies 3 = -3B \implies B = -1.$$

So a partial fraction decomposition is

$$\frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2} = 2x - 1 + \frac{2}{x-1} - \frac{1}{x+2}.$$

$$\begin{aligned}\int \frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2}\,dx &= \int \left[2x - 1 + \frac{2}{x-1} - \frac{1}{x+2}\right]\,dx \\ &= 2\int x\,dx - \int\,dx + 2\int \frac{dx}{x-1} - \int \frac{dx}{x+2} \\ &= x^2 - x + 2\ln|x-1| - \ln|x+2| + C.\end{aligned}$$

Example. Show that $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$ for $a \neq 0$.

Solution. Note that $x^2 - a^2 = (x-a)(x+a)$. For a partial fraction decomposition, let

$$\frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}.$$

Multiplying by $(x-a)(x+a)$, we get

$$1 = A(x+a) + B(x-a).$$

Plugging $x = a$, we get

$$1 = 2aA \implies A = \frac{1}{2a}.$$

Plugging $x = -a$, we get

$$1 = -2aB \implies B = -\frac{1}{2a}.$$

So a partial fraction decomposition is

$$\frac{1}{(x-a)(x+a)} = \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

$$\begin{aligned}\int \frac{dx}{x^2 - a^2} &= \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)\,dx \\ &= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.\end{aligned}$$

2. $Q(x)$ is a product of repeated linear factors, i.e.,

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)^{n_1}(a_2x+b_2)^{n_2}\cdots(a_kx+b_k)^{n_k}}.$$

In this case, $(a_ix+b_i)^{n_i}$ contributes the following to the partial fraction decomposition:

$$\frac{A_1}{(a_ix+b_i)} + \frac{A_2}{(a_ix+b_i)^2} + \cdots + \frac{A_{n_i}}{(a_ix+b_i)^{n_i}}.$$

Example. Evaluate $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx.$

Solution. For a partial fraction decomposition, let

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Multiplying by $(2x+1)(x-2)^2$, we get

$$x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1) \qquad (2)$$

Plugging $x=-\frac{1}{2}$ in (2), we get

$$\begin{aligned} \left(-\tfrac{1}{2}\right)^2+5\cdot\tfrac{1}{2}+16 &= A\left(-\tfrac{1}{2}-2\right)^2+B\left(-\tfrac{1}{2}-2\right)\left(-2\cdot\tfrac{1}{2}+1\right)+C\left(-2\cdot\tfrac{1}{2}+1\right)\\ \Longrightarrow \quad \tfrac{75}{4} &= \tfrac{25}{4}A\\ \Longrightarrow \quad A &= 3. \end{aligned}$$

Plugging $x=2$ in (2), we get

$$2^2-5\cdot2+16 = A(2-2)^2+B(2-2)(2\cdot2+1)+C(2\cdot2+1) \implies 10=5C \implies C=2.$$

There are two ways to get B:

(a) Plug some value of $x \neq -\frac{1}{2}, 2$.
Plugging $x=0$ in (2), we get

$$0^2-5\cdot0+16 = 3(0-2)^2+B(0-2)(2\cdot0+1)+2(2\cdot0+1) \implies 16=14-2B \implies B=-1.$$

(b) Compare the coefficients of like powers of x in both sides of (2).
Rewriting (2), we get

$$\begin{aligned} x^2-5x+16 &= 3(x^2-4x+4)+B(2x^2-3x-2)+2(2x+1)\\ &= (3+2B)x^2+(-12-3B+4)x+(12-2B+2) \end{aligned}$$

Comparing the coefficient of x^2 in both sides, we get

$$1 = 3+2B \implies B=-1.$$

So a partial fraction decomposition is

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{3}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}.$$

$$\begin{aligned}\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx &= 3\int \frac{dx}{2x+1} - \int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2} \\ &= \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C.\end{aligned}$$

3. $Q(x)$ contains an irreducible quadratic factor ax^2+bx+c (which cannot be written as the product of two linear factors because $b^2-4ac<0$).

   In this case, ax^2+bx+c contributes the following to the partial fraction decomposition:

$$\frac{A_1x+B_1}{ax^2+bx+c}.$$

A relevant formula:

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C,\ a \neq 0}$$

Example. Evaluate $\displaystyle\int \frac{x^2+2x-1}{x^3+x^2+x}\,dx$.

Solution. First note that $x^3+x^2+x = x(x^2+x+1)$ where x^2+x+1 is irreducible. For a partial fraction decomposition, let

$$\frac{x^2+2x-1}{x^3+x^2+x} = \frac{x^2+2x-1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}.$$

Multiplying by $x(x^2+x+1)$, we get

$$x^2+2x-1 = A(x^2+x+1) + (Bx+C)x = (A+B)x^2 + (A+C)x + A.$$

Comparing the coefficients of like powers of x, we get

$$A+B=1,\ A+C=2,\ A=-1.$$

So $A=-1$, $B=1-A=2$, and $C=2-A=3$. So a partial fraction decomposition is

$$\frac{x^2+2x-1}{x^3+x^2+x} = -\frac{1}{x} + \frac{2x+3}{x^2+x+1}.$$

To integrate the second fraction, we complete the square in the denominator:

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\begin{aligned}
\int \frac{x^2 + 2x - 1}{x^3 + x^2 + x}\,dx &= \int \left(-\frac{1}{x} + \frac{2x+3}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\right) dx \\
&= -\ln|x| + \int \frac{2x+3}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx \\
&= -\ln|x| + \int \frac{2(u - \frac{1}{2}) + 3}{u^2 + \frac{3}{4}}\,du \quad \left(\text{Let } u = x + \frac{1}{2}. \text{ Then } du = dx\right) \\
&= -\ln|x| + \int \frac{2u+2}{u^2 + \frac{3}{4}}\,du \\
&= -\ln|x| + \int \frac{2u}{u^2 + \frac{3}{4}}\,du + 2\int \frac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= -\ln|x| + \ln\left|u^2 + \frac{3}{4}\right| + 2\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u}{\frac{\sqrt{3}}{2}}\right) + C \\
&= -\ln|x| + \ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + C \\
&= -\ln|x| + \ln\left|x^2 + x + 1\right| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

4. $Q(x)$ contains a repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, $n \geq 2$.

   In this case, $(ax^2+bx+c)^n$ contributes the following to the partial fraction decomposition:

$$\frac{A_1x + B_1}{(ax^2 + bx + c)} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

Example. Evaluate $\displaystyle\int \frac{dx}{x(x^2+4)^2}$.

Solution. For a partial fraction decomposition, let

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2+4)^2}.$$

Multiplying by $x(x^2+4)^2$, we get

$$\begin{aligned}
1 &= A(x^2+4)^2 + (Bx + C)x(x^2+4) + (Dx + E)x \\
&= A(x^4 + 8x^2 + 16) + (Bx^4 + Cx^3 + 4Bx^2 + 4Cx) + (Dx^2 + Ex) \\
&= (A + B)x^4 + Cx^3 + (8A + 4B + D)x^2 + (4C + E)x + 16A.
\end{aligned}$$

Comparing the coefficients of like powers of x, we get

$$A+B=0,\ C=0,\ 8A+4B+D=0,\ 4C+E=0,\ 16A=1.$$

So $A=\frac{1}{16}$, $B=-A=-\frac{1}{16}$, $C=0$, $D=-8A-4B=-\frac{1}{4}$, and $E=-4C=0$. So a partial fraction decomposition is

$$\frac{1}{x(x^2+4)^2}=\frac{1}{16x}-\frac{x}{16(x^2+4)}-\frac{x}{4(x^2+4)^2}.$$

$$\begin{aligned}\int\frac{dx}{x(x^2+4)^2}&=\int\left(\frac{1}{16x}-\frac{x}{16(x^2+4)}-\frac{x}{4(x^2+4)^2}\right)dx\\&=\frac{1}{16}\int\frac{dx}{x}-\frac{1}{32}\int\frac{2x\,dx}{x^2+4}-\frac{1}{8}\int\frac{2x\,dx}{(x^2+4)^2}\\&=\frac{1}{16}\ln|x|-\frac{1}{32}\ln|x^2+4|+\frac{1}{8(x^2+4)}+C\end{aligned}$$

Exercises

Evaluate the following integrals by Partial Fractions:

1. $\displaystyle\int\frac{x^3-4x-10}{x^2-x-6}\,dx$
2. $\displaystyle\int\frac{x^2+1}{(x-3)(x-2)^2}\,dx$
3. $\displaystyle\int\frac{4x^2-21x+17}{(x-3)^3}\,dx$
4. $\displaystyle\int\frac{4x}{x^3+x^2+x+1}\,dx$
5. $\displaystyle\int\frac{2x^2+15x-16}{3x^3-4x^2+6x-8}\,dx$
6. $\displaystyle\int\frac{12x^3+9x^2-56x-145}{(x^2+10x+25)(3x^2+1)}\,dx$
7. $\displaystyle\int\frac{x^4-6x^3+19x^2-49x+135}{(x-1)(x^2+9)^2}\,dx$
8. $\displaystyle\int\frac{-x^5+10x^4+46x^2+19x+120}{(x^2-2x+5)(x^2+6)^2}\,dx$

Answers

1. $\dfrac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| + C$

2. $10\ln|x-3| - 9\ln|x-2| + \dfrac{5}{x-2} + C$

3. $4\ln|x-3| - \dfrac{3}{(x-3)} + \dfrac{5}{(x-3)^2} + C$

4. $-2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C$

5. $\dfrac{2}{3}\ln|3x-4| + \dfrac{5}{\sqrt{2}}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right) + C$

6. $4\ln|x+5| + \dfrac{15}{(x+5)} - 2\sqrt{3}\tan^{-1}(\sqrt{3}x) + C$

7. $\ln|x-1| - 2\tan^{-1}\left(\dfrac{x}{3}\right) + \dfrac{5}{2(x^2+9)} + C$

8. $\ln|x^2 - 2x + 5| + \tan^{-1}\left(\dfrac{x-1}{2}\right) - \dfrac{3}{2}\ln|x^2+6| + \dfrac{2\sqrt{6}}{3}\tan^{-1}\left(\dfrac{x}{\sqrt{6}}\right) - \dfrac{17}{2(x^2+6)} + C$

2.6 Improper Integrals

An integral $\displaystyle\int_a^b f(x)\,dx$ is called an *improper integral* if

1. $a = -\infty$ or $b = \infty$, i.e., the interval of integration is an infinite interval, or,

2. f has an infinite discontinuity at some point c in $[a, b]$ which means $x = c$ is a vertical asymptote of the curve $y = f(x)$, i.e.,

$$\lim_{x\to c^+} f(x) = \pm\infty \text{ or } \lim_{x\to c^-} f(x) = \pm\infty.$$

Infinite Intervals:

1. If $\displaystyle\int_a^t f(x)\,dx$ exists for all $t \geq a$, then we define $\displaystyle\int_a^\infty f(x)\,dx$ as the following limit when it exists:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

2. If $\displaystyle\int_t^a f(x)\,dx$ exists for all $t \leq a$, then we define $\displaystyle\int_{-\infty}^a f(x)\,dx$ as the following limit when it exists:

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

3. We define $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ as

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx,$$

provided both $\displaystyle\int_{-\infty}^{a} f(x)\,dx$ and $\displaystyle\int_{a}^{\infty} f(x)\,dx$ exist.

An improper integral is *convergent* if the corresponding limit exists. Otherwise it is *divergent*.

Example. Evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{x^2+1}$ and $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$.

Solution.

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{x^2+1} &= \lim_{t\to\infty}\int_{0}^{t} \frac{dx}{x^2+1}\\ &= \lim_{t\to\infty} \tan^{-1} x\Big|_0^t\\ &= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right)\\ &= \lim_{t\to\infty} \tan^{-1} t\\ &= \frac{\pi}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{x^2+1} &= \lim_{t\to-\infty}\int_{t}^{0} \frac{dx}{x^2+1}\\ &= \lim_{t\to-\infty} \tan^{-1} x\Big|_t^0\\ &= \lim_{t\to-\infty} \left(-\tan^{-1} t\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

Note that the preceding integral can also be obtained by observing the symmetry of the function $f(x) = \dfrac{1}{x^2+1}$. Since both $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2+1}$ and $\displaystyle\int_{0}^{\infty} \frac{dx}{x^2+1}$ are convergent,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Example. For $a > 0$, show that $\displaystyle\int_{a}^{\infty} \frac{1}{x^p}\,dx = \frac{a^{1-p}}{p-1}$ if $p > 1$ and the integral is divergent if $p \le 1$.

Solution. First note that

$$\int \frac{1}{x^p}\,dx = \begin{cases} \frac{x^{1-p}}{1-p} & \text{if } p \neq 1 \\ \ln x & \text{if } p = 1. \end{cases}$$

If $p = 1$, then

$$\int_a^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big|_a^t = \lim_{t\to\infty} (\ln t - \ln a) = \infty.$$

If $p < 1$, then

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_a^t = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right) = \infty \text{ (as } 1-p > 0).$$

Thus the integral is divergent if $p \leq 1$. If $p > 1$, then

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_a^t = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right) = 0 - \frac{a^{1-p}}{1-p} \text{ (as } 1-p < 0).$$

Infinite Discontinuities:

1. If f is continuous on $[a, b)$ and f has an infinite discontinuity at b, i.e., $\lim_{x\to b^-} f(x) = \pm\infty$, then we define $\int_a^b f(x)\,dx$ as follow:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2. If f is continuous on $(a, b]$ and f has an infinite discontinuity at a, i.e., $\lim_{x\to a^+} f(x) = \pm\infty$, then we define $\int_a^b f(x)\,dx$ as follow:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

3. If f has an infinite discontinuity at c, $a < c < b$, then we define $\int_a^b f(x)\,dx$ as follow:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist.

An improper integral is *convergent* if the corresponding limits exist. Otherwise it is *divergent*.

Example. Show that $\int_0^1 \ln x\,dx = -1$.

Solution. First note that ln is continuous on $(0,1]$ and ln has an infinite discontinuity at 0 as $\lim_{x\to 0^+} \ln x = -\infty$.

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{t\to 0^+} \int_t^1 \ln x\,dx \\ &= \lim_{t\to 0^+} x\ln x - x\Big|_t^1 \\ &= \lim_{t\to 0^+} (-1 - t\ln t + t) \\ &= -1 - \lim_{t\to 0^+} t\ln t \\ &= -1 - \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \ \left(\frac{\infty}{\infty}\right) \\ &= -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} \ \text{(by L'Hospital's Rule)} \\ &= -1 + \lim_{t\to 0^+} t \\ &= -1.\end{aligned}$$

Example. Show that $\int_{-1}^1 \frac{dx}{x}$ is divergent.

Solution. First note that the fundamental theorem of calculus is not applicable here as the integrand is not defined at 0. Consequently, the following is incorrect:

$$\int_{-1}^1 \frac{dx}{x} = \ln|x|\Big|_{-1}^1 = \ln|1| - \ln|-1| = 0$$

Note that $\frac{1}{x}$ is continuous on $[-1,0)\cup(0,1]$ and it has an infinite discontinuity at 0 as $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} \\ &= \lim_{t\to 0^+} \ln|x|\Big|_t^1 \\ &= \lim_{t\to 0^+} (0 - \ln|t|) \\ &= \infty.\end{aligned}$$

Then $\int_0^1 \frac{dx}{x}$ is divergent and consequently $\int_{-1}^1 \frac{dx}{x}$ is divergent.

Comparison Test:
Suppose that f and g are continuous and $0 \le g(x) \le f(x)$ for all $x \ge a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is also convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is also divergent.

Example. Use the Comparison Test to show that $\int_1^\infty \frac{dx}{x^2+e^{-x}}$ is convergent.

Solution. For all $x \geq 1$,

$$0 \leq \frac{1}{x^2+e^{-x}} \leq \frac{1}{x^2}.$$

Now $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by the integral p-test where $p = 2 > 1$. By the Comparison Test, $\int_1^\infty \frac{dx}{x^2+e^{-x}}$ is convergent.

Now we will see in the following how the choice of a bigger function matters: for all $x \geq 1$,

$$0 \leq \frac{1}{x^2+e^{-x}} \leq \frac{1}{e^{-x}}.$$

Now $\int_1^\infty \frac{1}{e^{-x}}\,dx = \int_1^\infty e^x\,dx$ is divergent. Then we cannot apply the Comparison Test.

Exercises

1. Determine if each of the following improper integrals converges and, if so, evaluate it.
   (a) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$
   (b) $\int_{-\infty}^0 x^2e^{-x^3}\,dx$
   (c) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$
   (d) $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$
   (e) $\int_{-2}^3 \frac{1}{x^4}\,dx$
2. Determine if each of the following improper integrals is convergent or divergent.
   (a) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$, (b) $\int_1^\infty \frac{\sin^2 x}{x^2+1}\,dx$.

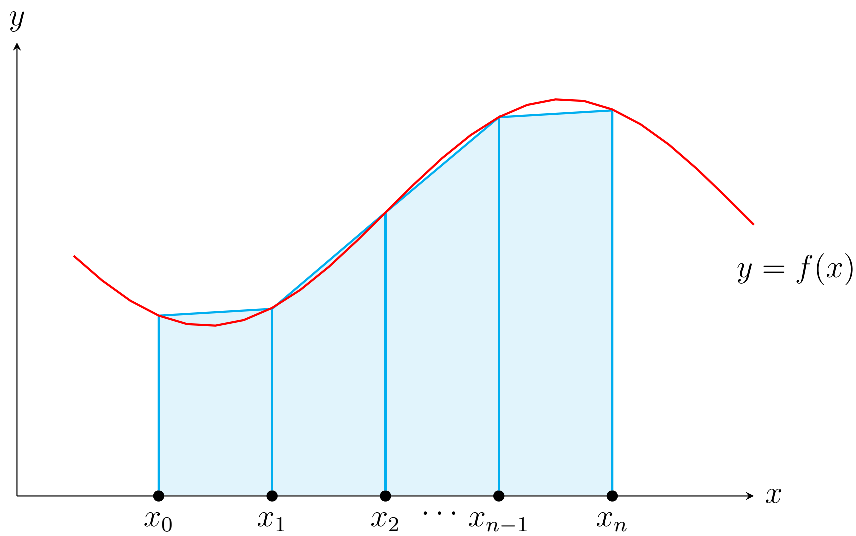
Answers

1. (a) 1, (b) diverges to ∞, (c) π, (d) 2, (e) divergent.
2. (a) divergent, (b) convergent.

2.7 Numerical Integration

Sometimes it is hard to calculate a definite integral analytically. For example, $\int_0^1 e^{x^2}\,dx$. To approximate such an integral we break $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b-a)/n$. Then we approximate the integral by a finite sum given by a *quadrature formula*:

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{i=1}^{n} c_i f(x_i^*) \text{ for some } c_i \text{ where } x_i^* \in [x_{i-1}, x_i]$$



The **Midpoint Rule** is

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right).$$

Note that the RHS is the sum of areas of n rectangles on subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ where the height of each rectangle is the height of the curve at the midpoint of the corresponding subinterval.

The **Trapezoidal Rule** is

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\Big].$$

Note that the RHS is the sum of areas of n trapezoids on subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

Example. Approximate $\int_0^2 e^x\,dx$ using 4 subintervals in (a) Trapezoidal Rule, (b) Midpoint Rule.

Solution. First of all let's find the exact integral: $\int_0^2 e^x\,dx = e^x\Big]_0^2 = e^2 - 1 \approx 6.389$.
$n = 4 \implies h = (2-0)/4 = 0.5$ and the 4 subintervals are $[0, 0.5],\ [0.5, 1],\ [1, 1.5],\ [1.5, 2]$.

$$\text{Trapezoidal Rule}: \int_0^2 e^x\,dx \approx \frac{0.5}{2}\Big[e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2\Big] = 6.52$$

$$\text{Midpoint Rule}: \int_0^2 e^x\,dx \approx 0.5\Big[e^{0.25} + e^{0.75} + e^{1.25} + e^{1.75}\Big] = 6.32$$

Exercises

Use (a) the Midpoint Rule and (b) the Trapezoidal Rule to approximate the following integrals (use 2 decimal places):

1. $\int_0^1 e^{x^2}\,dx$, $n = 4$

2. $\int_1^4 \sqrt{\ln x}\,dx$, $n = 6$

Answers

1. (a) 1.44 (b) 1.49

2. (a) 2.68 (b) 2.59

Additional Exercises

Evaluate the following integrals:

1. $\int \sin^2 x \cos x \ln(\sin x)\, dx$ (Hint. substitute $z = \sin x$, then by parts),
2. $\int x \sin(x^2) \cos(3x^2)\, dx$ (Hint. substitute $u = x^2$, then trig integral),
3. $\int \frac{\sqrt{4 + (\ln x)^2}}{x}\, dx$ (Hint. substitute $u = \ln x$, then trig substitution, then by parts),
4. $\int \frac{1}{x\sqrt{4x+1}}\, dx$ (Hint. substitute $u = \sqrt{4x+1}$, then partial fractions),

Answers

1. $\frac{1}{9} \sin^3 x\, [3 \ln(\sin x) - 1] + C$
2. $\frac{1}{8} \cos(2x^2) - \frac{1}{16} \cos(4x^2) + C$
3. $\frac{1}{2} \ln x \sqrt{4 + (\ln x)^2} + 2 \ln \left[\ln x + \sqrt{4 + (\ln x)^2}\right] + C$
4. $\ln \left| \frac{\sqrt{4x+1} - 1}{\sqrt{4x+1} + 1} \right| + C$

Chapter 3

Applications of the Integrals

In this chapter we express various quantities such as area and volume as integrals. The basic idea is to approximate such a quantity by a Riemann sum using n subintervals of an interval. Then the approximation approaches the quantity as n approaches ∞.

3.1 Average Value of a Function

If f is a continuous function on $[a, b]$, then the average value f_{ave} of f over $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Proof. Break $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b-a)/n$. Choose x_i^* from $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. The average value of $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This approximation of f_{ave} gets better as $n \to \infty$. Thus

$$f_{ave} = \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

□

The Mean Value Theorem for Integrals: If f is a continuous function on $[a, b]$, then there is at least one number c in $[a, b]$ such that $f(c) = f_{ave}$, i.e.,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. Find the average value f_{ave} of $f(x) = x^2 - 2x + 5$ over $[1, 4]$ and find c in $[1, 4]$ such that $f(c) = f_{ave}$.

Solution.

$$\begin{aligned} f_{ave} &= \frac{1}{4-1}\int_1^4 (x^2 - 2x + 5)\,dx \\ &= \frac{1}{3}\left(\frac{x^3}{3} - x^2 + 5x\right)\Big|_1^4 \\ &= \frac{1}{3}\left[\left(\frac{64}{3} - 16 + 20\right) - \left(\frac{1}{3} - 1 + 5\right)\right] \\ &= \frac{21}{3} \\ &= 7. \end{aligned}$$

$$f(c) = f_{ave} \implies c^2 - 2c + 5 = 7 \implies c^2 - 2c - 2 = 0 \implies c = 1 \pm \sqrt{3}.$$

Since $1 + \sqrt{3}$ is in $[1, 4]$, $c = 1 + \sqrt{3}$.

Exercises

1. Consider $f(x) = \ln x$ over $[1, e]$.
   (a) Find the average value f_{ave} of f on $[1, e]$.
   (b) Find c in $[1, e]$ such that $f(c) = f_{ave}$ (such c exists by the MVT).
2. Consider $f(x) = \cos x$ over $\left[0, \frac{\pi}{2}\right]$.
   (a) Find the average value f_{ave} of f on $\left[0, \frac{\pi}{2}\right]$.
   (b) Find c in $\left[0, \frac{\pi}{2}\right]$ such that $f(c) = f_{ave}$.
3. Find b such that the average value of $f(x) = 3x^2 - 4x + 2$ on $[0, b]$ is 3.

Answers

1. (a) $\dfrac{1}{e-1}$, (b) $e^{\frac{1}{e-1}}$
2. (a) $\dfrac{2}{\pi}$, (b) $\cos^{-1}\left(\dfrac{2}{\pi}\right)$
3. $1 + \sqrt{2}$

3.2 Area Between Curves

If f and g are continuous functions on $[a, b]$, then the area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$ is

$$\int_a^b |f(x) - g(x)|\, dx.$$

In particular, if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area is

$$\int_a^b [f(x) - g(x)]\, dx.$$

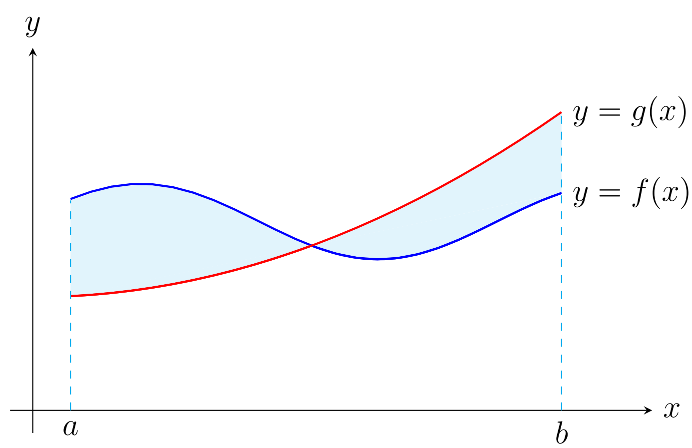
Proof. Break $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b - a)/n$. Choose x_i^* from $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$. Partition R into n regions $R_1, R_2, \ldots, R_n$ by intersecting R by the vertical lines at $x_1, x_2, \ldots, x_{n-1}$. Then the area of R_i is approximately $|f(x_i^*) - g(x_i^*)|\Delta x$. Therefore the area of R is

$$A(R) \approx \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\Delta x.$$

This approximation of $A(R)$ gets better as $n \to \infty$. Thus

$$A(R) = \lim_{n\to\infty} \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\Delta x = \int_a^b |f(x) - g(x)|\, dx.$$

□



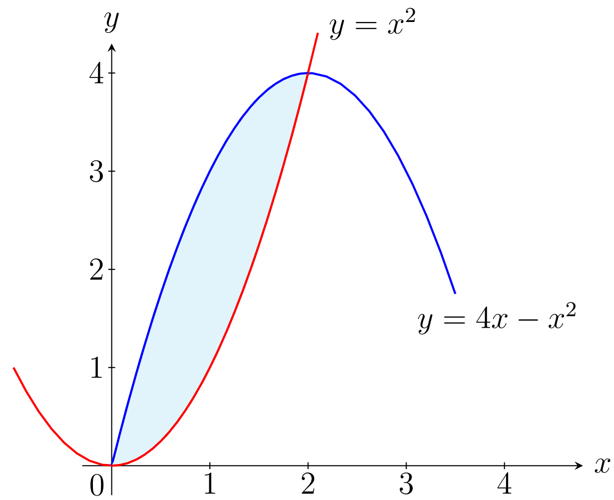
Note that $|f(x) - g(x)| = y_{\text{top}} - y_{\text{bottom}}$ on each subinterval of $[a, b]$.

Example. Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x - x^2$.

Solution. At the points of intersection of $y = x^2$ and $y = 4x - x^2$,

$$x^2 = 4x - x^2 \implies 2x^2 - 4x = 0 \implies 2x(x-2) = 0 \implies x = 0, 2.$$

So the points of intersection are $(0,0)$ and $(2,4)$.



The area of the region enclosed by $y = x^2$ and $y = 4x - x^2$ is

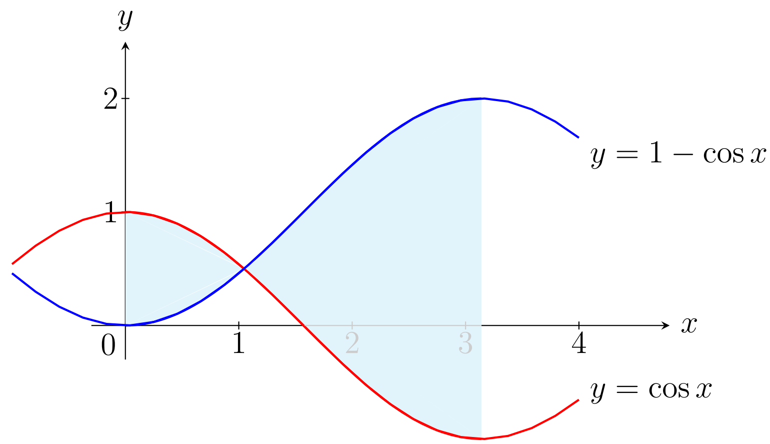
$$\begin{aligned}
\int_0^2 \left|(4x - x^2) - x^2\right| \, dx &= \int_0^2 \left(4x - x^2 - x^2\right) \, dx \\
&= \int_0^2 \left(4x - 2x^2\right) \, dx \\
&= \left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^2 \\
&= 2 \cdot 2^2 - \frac{2}{3}2^3 - 0 \\
&= \frac{8}{3}.
\end{aligned}$$

Example. Find the area of the region bounded by the curves $y = \cos x$ and $y = 1 - \cos x$ on $[0, \pi]$.

Solution. At the points of intersection of $y = \cos x$ and $y = 1 - \cos x$ on $[0, \pi]$,

$$\cos x = 1 - \cos x \implies 2\cos x = 1 \implies \cos x = \frac{1}{2} \implies x = \frac{\pi}{3}.$$

So the point of intersection is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.



Note that $\cos x \geq 1 - \cos x$ on $\left[0, \frac{\pi}{3}\right]$ and $\cos x \leq 1 - \cos x$ on $\left[\frac{\pi}{3}, \pi\right]$. The area of the region enclosed by $y = \cos x$ and $y = 1 - \cos x$ on $[0, \pi]$ is

$$\begin{aligned}
&\int_0^{\pi} |(1 - \cos x) - \cos x| \; dx \\
&= \int_0^{\frac{\pi}{3}} |(1 - \cos x) - \cos x| \; dx + \int_{\frac{\pi}{3}}^{\pi} |(1 - \cos x) - \cos x| \; dx \\
&= \int_0^{\frac{\pi}{3}} [-(1 - \cos x) + \cos x] \; dx + \int_{\frac{\pi}{3}}^{\pi} [(1 - \cos x) - \cos x] \; dx \\
&= \int_0^{\frac{\pi}{3}} [2\cos x - 1] \; dx + \int_{\frac{\pi}{3}}^{\pi} [1 - 2\cos x] \; dx \\
&= (2\sin x - x)\Big|_0^{\frac{\pi}{3}} + (x - 2\sin x)\Big|_{\frac{\pi}{3}}^{\pi} \\
&= \left[\left(2\sin\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right) - (2\sin 0 - 0)\right] + \left[(\pi - 2\sin\pi) - \left(\frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right)\right)\right] \\
&= \left[2\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right] + \left[\pi - \frac{\pi}{3} + 2\frac{\sqrt{3}}{2}\right] \\
&= 2\sqrt{3} + \frac{\pi}{3}.
\end{aligned}$$

If f and g are continuous functions on $[c, d]$, then the area of the region bounded by the curves $x = f(y)$, $x = g(y)$, and the horizontal lines $y = c$, $y = d$ is

$$\int_c^d |f(y) - g(y)| \; dy.$$

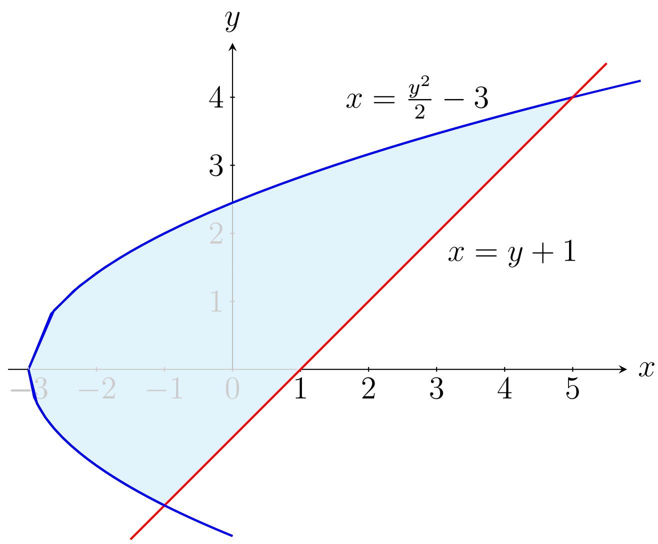
Note that $|f(y) - g(y)| = x_{\text{right}} - x_{\text{left}}$ on each subinterval of $[c, d]$.

Example. Find the area of the region enclosed by the curves $y^2 = 2x + 6$ and $y = x - 1$.

Solution. At the points of intersection of $y^2 = 2x + 6$ and $y = x - 1$,

$$2x + 6 = (x-1)^2 \implies x^2 - 4x - 5 = 0 \implies x = -1, 5.$$

So the points of intersection are $(-1, -2)$ and $(5, 4)$.



The area of the region enclosed by $x = \frac{y^2}{2} - 3$ and $x = y + 1$ is

$$\begin{aligned}
\int_{-2}^{4} (x_{\text{right}} - x_{\text{left}})\ dy &= \int_{-2}^{4} \left[(y+1) - \left(\frac{y^2}{2} - 3\right)\right]\ dy \\
&= \int_{-2}^{4} \left[-\frac{y^2}{2} + y + 4\right]\ dy \\
&= \left(-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right)\Bigg|_{-2}^{4} \\
&= \left(-\frac{4^3}{6} + \frac{4^2}{2} + 4\cdot 4\right) - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4(-2)\right) \\
&= 18.
\end{aligned}$$

Exercises

1. Find the area bounded by the curves $y = (x-2)^2$ and $y = x$.
2. Find the area bounded by the curves $y = \cos x$, $y = \sin(2x)$, $x = \frac{\pi}{2}$, and y-axis.
3. Find the area bounded by the curves $f(x) = x^3 + x^2 - 8x$ and $g(x) = x + 9$.
4. Find the area bounded by the parabolas $y^2 = \frac{x}{2}$ and $y^2 = x - 4$.

5. Find the area bounded by the curves $y^2 = 12 - 4x$ and $y = x$.

Answers

1. $\frac{9}{2}$
2. $\frac{1}{2}$
3. $\frac{148}{3}$
4. $\frac{32}{3}$
5. $\frac{64}{3}$

3.3 Volume of a Solid

Let S be the solid between the planes $x = a$ and $x = b$ such that $A(x)$ is the area of its vertical cross-section at x. If A is a continuous function on $[a, b]$, then the volume of S is

$$\int_a^b A(x)\, dx.$$

Proof. Break $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b - a)/n$. Choose x_i^* from $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Partition S into n solids $S_1, S_2, \ldots, S_n$ by intersecting S by the vertical planes at $x_1, x_2, \ldots, x_{n-1}$. Then the volume of S_i is approximately $A(x_i^*)\Delta x$. Therefore the volume of S is

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

This approximation of $V(S)$ gets better as $n \to \infty$. Thus

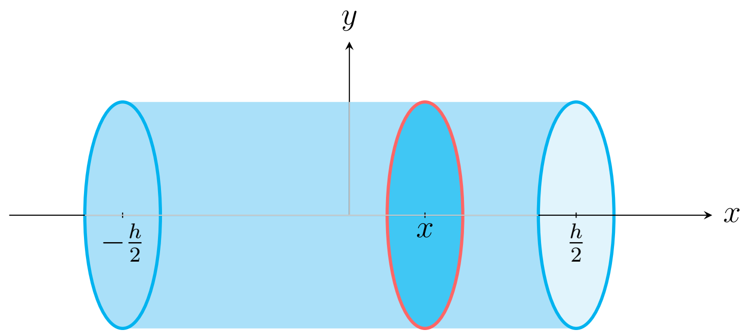
$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$

□

Example. Prove that the volume of a right circular cylinder with radius r and height h is $\pi r^2 h$.

Solution. Consider the right circular cylinder with radius r that is centered at the origin $(0, 0, 0)$ and lies on $\left[-\frac{h}{2}, \frac{h}{2}\right]$ on the x-axis. Its vertical cross-section at x is a disk which has radius r and consequently the area $A(x) = \pi r^2$. Since the cylinder lies on $\left[-\frac{h}{2}, \frac{h}{2}\right]$ on the x-axis, its volume is

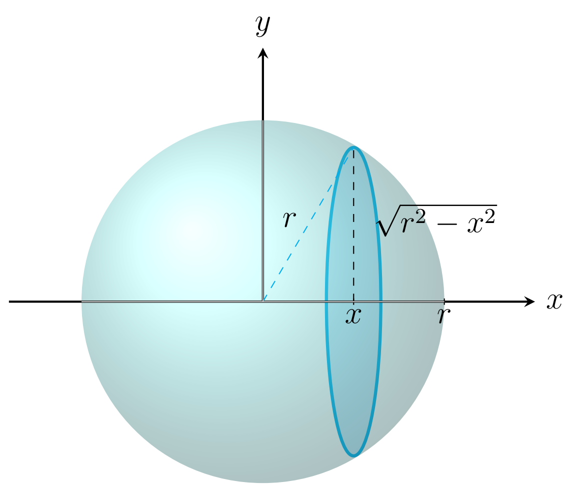
$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \pi r^2\, dx = \pi r^2 x\Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \pi r^2 \frac{h}{2} - \pi r^2 \frac{-h}{2} = \pi r^2 h.$$



Example. Prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution. Consider the sphere of radius r that is centered at the origin $(0, 0, 0)$. Its vertical cross-section at x is a disk which has radius $\sqrt{r^2 - x^2}$ and consequently the area $A(x) = \pi(\sqrt{r^2 - x^2})^2 = \pi(r^2 - x^2)$. Since the sphere lies on $[-r, r]$ on the x-axis, its volume is

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi \left(r^2 x - \frac{x^3}{3} \right)\Big|_{-r}^{r} = \pi \left(r^3 - \frac{r^3}{3} \right) - \pi \left(-r^3 + \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3.$$



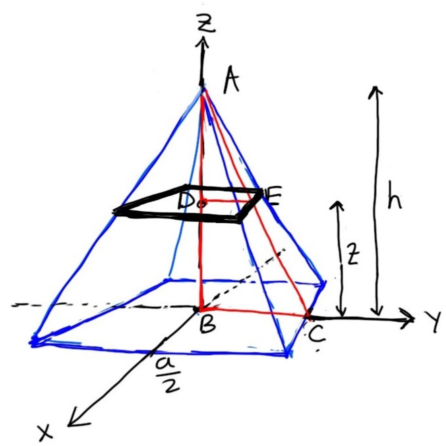
Example. Prove that the volume of a pyramid of height h whose base is a square of side a is $\frac{1}{3}a^2 h$.

Solution. Consider the square pyramid with height h whose base is on the xy-plane centered at the origin $(0, 0, 0)$. Its horizontal cross-section at z is square region with side, say s. By equating the ratio of the base and height of two similar triangles $\triangle ADE$ and $\triangle ABC$ (see figure), we have

$$\frac{s/2}{h - z} = \frac{a/2}{h} \implies s = \frac{a(h - z)}{h} = a\left(1 - \frac{z}{h}\right).$$

The horizontal cross-section at z has the area

$$A(z) = s^2 = a^2 \left(1 - \frac{z}{h}\right)^2 = a^2 \left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right).$$



Since the pyramid lies on $[0, h]$ on the z-axis, its volume is

$$\int_0^h a^2 \left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right)\, dz = a^2 \left(z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right)\Big|_0^h = a^2 \left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}a^2 h.$$

Exercises

1. Find the volume of the solid whose base is the semicircle $y = \sqrt{4 - x^2}$, $-2 \le x \le 2$, and the cross sections perpendicular to the x-axis are squares.
2. Find the volume of a right circular cone of height h whose base is a circle of radius r.
3. Find the volume of a pyramid with height h whose base is a square with side ℓ.

Answers

1. $\frac{32}{3}$
2. $\dfrac{1}{3}\pi r^2 h$
3. $\dfrac{1}{3}\ell^2 h$

3.4 Volume of a Solid of Revolution

A solid S of revolution is obtained by rotating a region about a line. First suppose S is obtained by rotating a region about the x-axis or a line on the xy-plane parallel to the x-axis. If S is between $x = a$ and $x = b$ such that $A(x)$ is the area of its vertical cross-section at x and A is a continuous function on $[a, b]$, then the volume of S is

$$V(S) = \int_a^b A(x)\, dx.$$

Disk Method: If the vertical cross-section is a disk with radius $r(x)$, then $A(x) = \pi r^2$ and

$$V(S) = \int_a^b \pi r^2 \, dx.$$

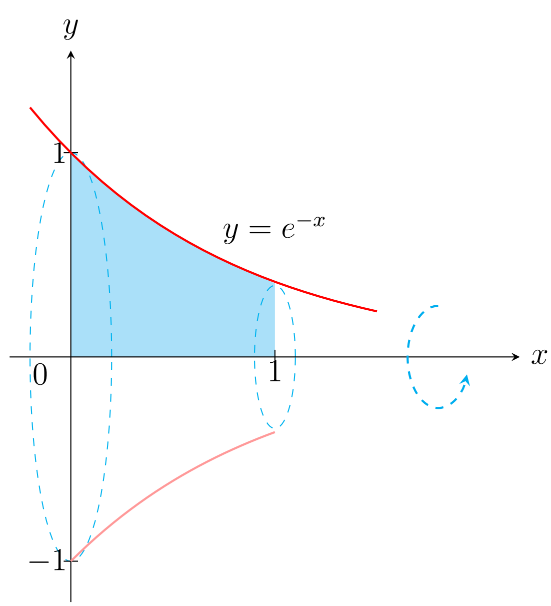
Washer Method: If the vertical cross-section is a washer (annular ring) with the inner radius $r_{in}(x)$ and the outer radius $r_{out}(x)$, then $A(x) = \pi \left(r_{out}^2 - r_{in}^2\right)$ and

$$V(S) = \int_a^b \pi \left(r_{out}^2 - r_{in}^2\right) \, dx.$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $x = 0$, $x = 1$, and $y = 0$ about the x-axis.

Solution. The cross-section of the solid at x perpendicular to the x-axis is a disk with the radius e^{-x} and the area $\pi \left(e^{-x}\right)^2 = \pi e^{-2x}$. Since the solid is in between $x = 0$ and $x = 1$, its volume is

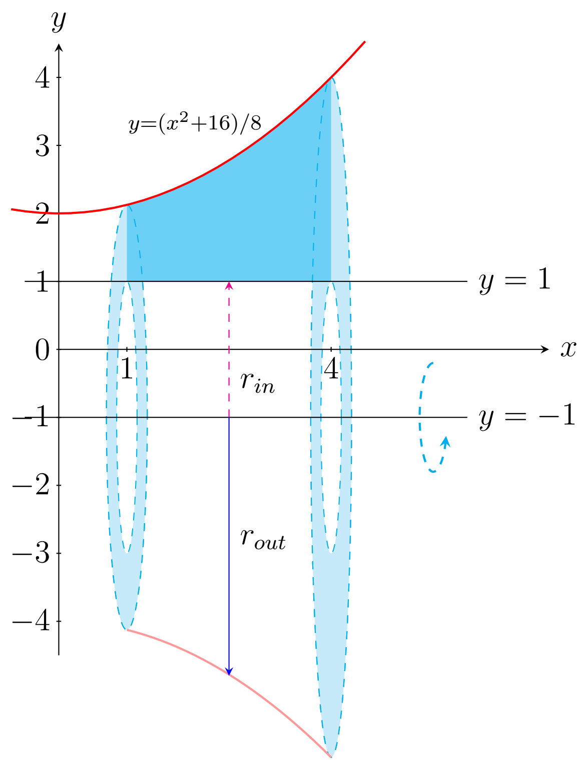
$$\int_0^1 \pi e^{-2x} \, dx = \pi \frac{e^{-2x}}{-2}\bigg|_0^1 = -\frac{\pi}{2}\left(e^{-2} - e^0\right) = \frac{\pi\left(e^2 - 1\right)}{2e^2}.$$



Example. Find the volume of the solid obtained by rotating the region bounded by $y = \frac{x^2+16}{8}$, $y = 1$, $x = 1$, and $x = 4$ about the line $y = -1$.

Solution. The cross-section of the solid at x perpendicular to the x-axis is a washer with the inner radius $r_{in} = 1 - (-1) = 2$ and the outer radius $r_{out} = \frac{x^2+16}{8} - (-1) = \frac{x^2}{8} + 3$. Since the solid is in between $x = 1$ and $x = 4$, its volume is

$$\begin{aligned}
\int_1^4 \pi \left(r_{out}^2 - r_{in}^2\right)\, dx &= \int_1^4 \pi \left(\left(\frac{x^2}{8} + 3 \right)^2 - 2^2 \right)\, dx \\
&= \int_1^4 \pi \left(\frac{x^4}{64} + \frac{3}{4}x^2 + 5 \right)\, dx \\
&= \int_1^4 \frac{\pi}{64} \left(x^4 + 48x^2 + 320\right)\, dx \\
&= \frac{\pi}{64} \left(\frac{x^5}{5} + 16x^3 + 320x \right)\Bigg|_1^4 \\
&= \frac{\pi}{64} \left[\left(\frac{4^5}{5} + 16 \cdot 4^3 + 320 \cdot 4 \right) - \left(\frac{1^5}{5} + 16 \cdot 1^3 + 320 \cdot 1 \right) \right] \\
&= \frac{10863\pi}{320}.
\end{aligned}$$



The above techniques will be similarly applied to a solid S obtained by rotating a region about the y-axis or a line on the xy-plane parallel to the y-axis:
If S is between $y = c$ and $y = d$ such that $A(y)$ is the area of its vertical cross-section at y and A is a continuous function on $[c, d]$, then the volume of S is

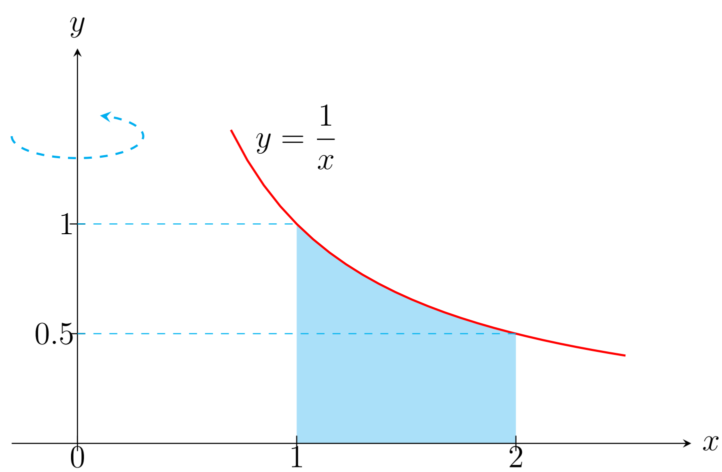
$$V(S) = \int_c^d A(y)\, dy.$$

We apply the disk method and the washer method similarly to find $A(y)$.

Example. Find the volume of the solid obtained by rotating the region bounded by $xy = 1$, $x = 1$, $x = 2$, and $y = 0$ about the y-axis.

Solution. The cross-section of the solid at y perpendicular to the y-axis is a washer with the inner radius $r_{in} = 1$ and the outer radius $r_{out} = 2$ when $0 \le y \le 0.5$ and $r_{out} = \frac{1}{y}$ when $0.5 \le y \le 1$. Since the solid is in between $y = 0$ and $y = 1$, its volume is

$$\begin{aligned}\int_0^1 \pi\left(r_{out}^2 - r_{in}^2\right)\, dy &= \int_0^{0.5} \pi\left(2^2 - 1^2\right)\, dy + \int_{0.5}^1 \pi\left(\left(\frac{1}{y}\right)^2 - 1^2\right)\, dy \\ &= \int_0^{0.5} 3\pi\, dy + \int_{0.5}^1 \pi\left(y^{-2} - 1\right)\, dy \\ &= 3\pi y\Big|_0^{0.5} + \pi\left(-\frac{1}{y} - y\right)\Big|_{0.5}^1 \\ &= 3\pi \cdot 0.5 + \pi\left[(-1-1) - \left(-\frac{1}{0.5} - 0.5\right)\right] \\ &= 1.5\pi + 0.5\pi \\ &= 2\pi.\end{aligned}$$



Exercises

Find the volume of the solids obtained by rotating the regions enclosed by the following graphs about the given axes.

1. $y = 1 - x^2$, $y = 0$ about the x-axis.

2. $y = \ln x$, $y = 1$, $y = 2$, $x = 0$ about the the y-axis.

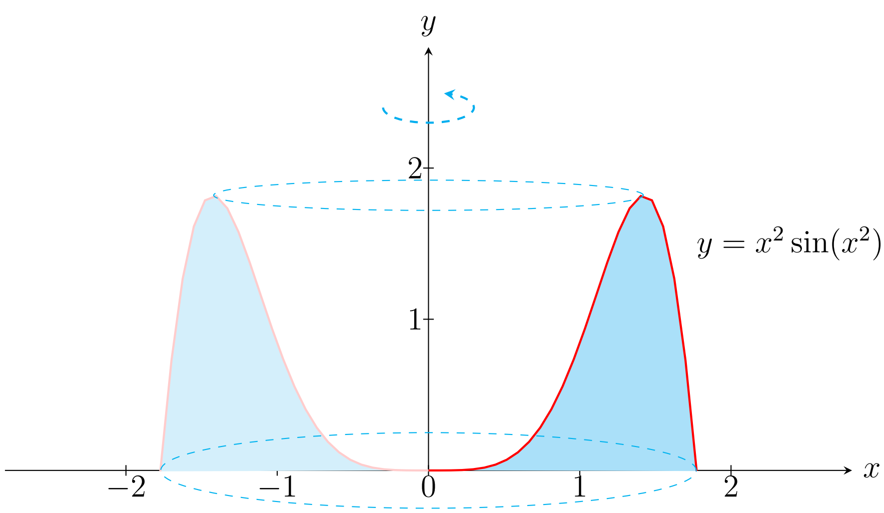
3. $y = x^3$, $x = x^6$ about the (a) x-axis, (b) y-axis.
4. $y = \sqrt{x}$, $y = x$ about $y = 1$.
5. $y = x^2$, $x = y^2$ about $x = -1$.

Answers

1. $\frac{16\pi}{15}$
2. $\frac{\pi(e^4 - e^2)}{2}$
3. (a) $\frac{6\pi}{91}$, (b) $\frac{3\pi}{20}$
4. $\frac{\pi}{6}$
5. $\frac{29\pi}{30}$

3.5 Cylindrical Shells Method

Consider the following problem:
Find the volume of the solid obtained by rotating the region bounded by $y = x^2 \sin(x^2)$, $x = 0$, $x = \sqrt{\pi}$, and $y = 0$ about the y-axis.



First note that the cross-section of the solid at y perpendicular to the y-axis is a washer. To apply the washer method, we need to find the inner radius and the outer radius which will require to solve for x in terms of y from $y = x^2 \sin(x^2)$. It is complicated and sometimes analytically impossible to solve for x in terms of y from $y = f(x)$. We have a better method for this kind of problems:

Cylindrical Shells Method: Consider the region bounded by $y = f(x)$, $x = a$, $x = b$, and the x-axis. If S is the solid obtained by rotating the region about the y-axis, then the volume of S is

$$V(S) = \int_a^b 2\pi x f(x)\, dx.$$

We can remember the preceding formula by associating $2\pi x$, $f(x)$, and dx with the circumference, height, and thickness of the cylindrical shell with the radius x respectively.

Proof. Break $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b-a)/n$. Partition S into n solids $S_1, S_2, \dots, S_n$ by intersecting S by the vertical planes at $x_1, x_2, \dots, x_{n-1}$. Then the volume of S_i is approximately that of the cylindrical shell with the inner radius x_{i-1}, the outer radius x_i, and height $f(x_i^*)$ where $x_i^* = \frac{x_{i-1}+x_i}{2}$, $i = 1, 2, \dots, n$. Therefore

$$\begin{aligned} V(S_i) &\approx \pi x_i^2 f(x_i^*) - \pi x_{i-1}^2 f(x_i^*) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\frac{x_{i-1}+x_i}{2}(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*) x_i^* \Delta x. \end{aligned}$$

Thus the volume of S is

$$V(S) \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x.$$

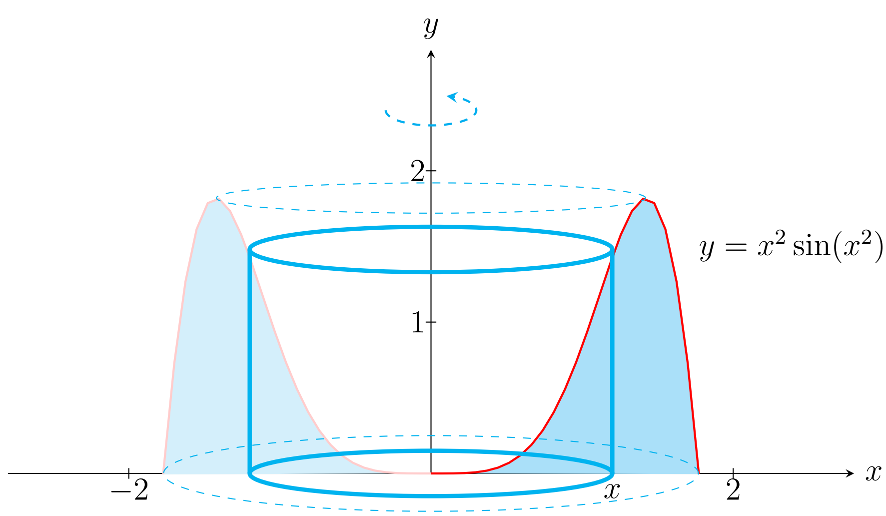
This approximation of $V(S)$ gets better as $n \to \infty$. Thus

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x)\, dx.$$

□

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^2\sin(x^2)$, $x = 0$, $x = \sqrt{\pi}$, and $y = 0$ about the y-axis.

Solution.



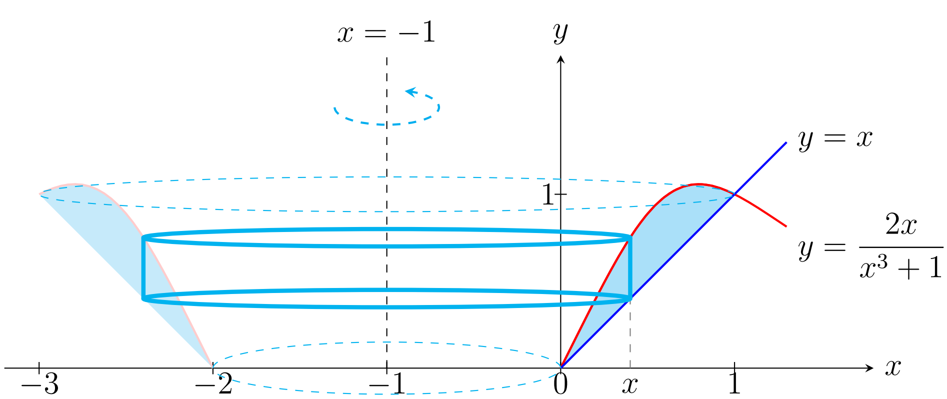
The volume of the solid is

$$\begin{aligned}
&\int_0^{\sqrt{\pi}} 2\pi x \cdot x^2 \sin(x^2)\, dx \\
&= \pi \int_0^{\sqrt{\pi}} x^2 \sin(x^2) \cdot 2x\, dx \\
&= \pi \int_0^{\pi} u \sin u\, du \quad \left(\text{Let } u = x^2. \text{ Then } du = 2x\, dx\right) \\
&= \pi \left(-u\cos u + \sin u\right)\Big|_0^{\pi} \quad \text{(Integrating by parts)} \\
&= \pi\left(-\pi\cos\pi + \sin\pi\right) - \pi\left(-0\cos 0 + \sin 0\right) \\
&= \pi^2.
\end{aligned}$$

For some problems, the formula of Cylindrical Shells Method $V(S) = \int_a^b 2\pi x f(x)\, dx$ is modified by replacing x and $f(x)$ by the radius and the height of the cylindrical shell at x respectively.

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = \frac{2x}{x^3+1}$ about the line $x = -1$.

Solution.



First note that $y = x$ and $y = \frac{2x}{x^3+1}$ intersect at $x = 0, 1$. The cylindrical shell at x has radius

$x-(-1)$ and height $\frac{2x}{x^3+1}-x$. The volume of the solid is

$$\begin{aligned}
&\int_0^1 2\pi(x+1)\left(\frac{2x}{x^3+1}-x\right)\,dx \\
&= 2\pi\int_0^1 \left[-(x+1)x+(x+1)\frac{2x}{x^3+1}\right]\,dx \\
&= 2\pi\int_0^1 \left[-x^2-x+\frac{2x}{x^2-x+1}\right]\,dx \\
&= 2\pi\int_0^1 \left[-x^2-x+\frac{2x-1}{x^2-x+1}+\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\,dx \\
&= 2\pi\left(-\frac{x^3}{3}-\frac{x^2}{2}+\ln|x^2-x+1|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right)\Bigg|_0^1 \\
&= 2\pi\left(-\frac{5}{6}+\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) \\
&= 2\pi\left(-\frac{5}{6}+\frac{2\pi}{3\sqrt{3}}\right).
\end{aligned}$$

The roles of x and y in the Cylindrical Shells Method may be switched for some problems.

Example. Find the volume of the solid obtained by rotating the region bounded by $x=y(3-e^y)$ and the y-axis about the x-axis.

Solution. First note that at the points of intersection of $x=y(3-e^y)$ and $x=0$, we have

$$y(3-e^y)=0 \implies y=0,\ln 3.$$

The cylindrical shell at y has radius y and height $y(3-e^y)$. The volume of the solid is

$$\begin{aligned}
&\int_0^{\ln 3} 2\pi y\cdot y(3-e^y)\,dy \\
&= 2\pi\int_0^{\ln 3}\left(3y^2-y^2e^y\right)\,dy \\
&= 2\pi\left(y^3-e^y(y^2-2y+2)\right)\Big|_0^{\ln 3} \quad \text{(Integrating by parts)} \\
&= 2\pi\left((\ln 3)^3-3((\ln 3)^2-2\ln 3+2)\right)+4\pi \\
&= 2\pi\left((\ln 3)^3-3(\ln 3)^2+6\ln 3-4)\right).
\end{aligned}$$

Exercises

Use the cylindrical shells method to find the volume of the solids obtained by rotating the regions enclosed by the following graphs about the given axes.

1. $y = \sin(x^2)$, $x = 0$, $x = \sqrt{\pi}$, $y = 0$ about the the y-axis.

2. $x = y(4 - y)$, $x = 0$ about the x-axis.

3. $y = x - x^2$, $y = 0$ about $x = 2$.

Answers

1. 2π
2. $\frac{128\pi}{3}$
3. $\frac{\pi}{2}$

3.6 Arc Length

Suppose f is a continuously differentiable function on $[a, b]$, i.e., f' is continuous on $[a, b]$. Then the length L of the curve $y = f(x)$ on $[a, b]$ is

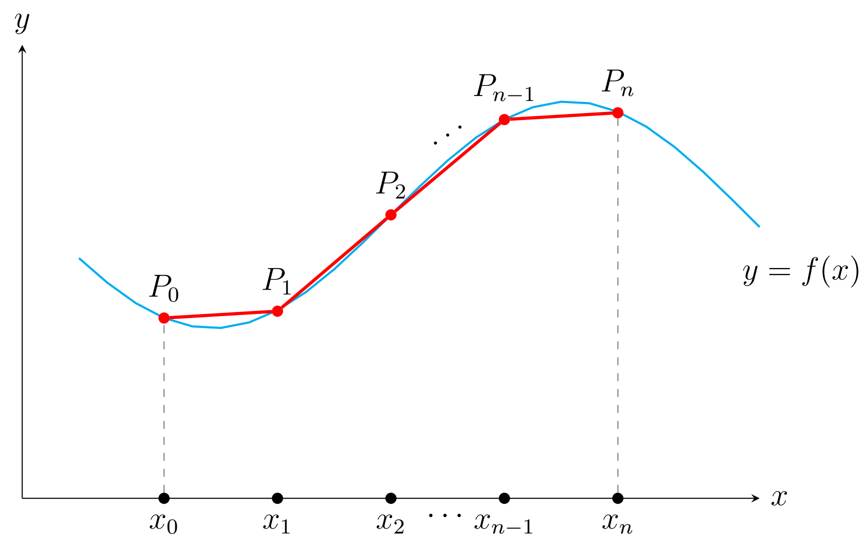
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Proof. Break $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $x_i = a + i\Delta x$ and $\Delta x = (b - a)/n$. Consider the $n + 1$ points on the curve $y = f(x)$:

$$P_0(x_0, y_0), P_1(x_1, y_1), \ldots, P_n(x_n, y_n),$$

where $y_i = f(x_i)$ for $i = 0, 1, 2, \ldots, n$. Note that $L \approx \sum_{i=1}^{n} |P_{i-1}P_i|$ where

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}.$$



By the Mean Value Theorem on f on $[x_{i-1}, x_i]$, we get

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x,$$

for some x_i^* in (x_{i-1}, x_i). Then

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2} = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Therefore the length L of the curve $y = f(x)$ on $[a, b]$ is

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This approximation of L gets better as $n \to \infty$. Thus

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

□

Example. Prove that the circumference of a circle of radius r is $2\pi r$.

Solution. The circumference of a circle of radius r is twice the arc length L of the semicircle $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$.

$$\begin{aligned}
L &= \int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx \\
&= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\
&= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx \\
&= r\sin^{-1}\left(\frac{x}{r}\right)\Big|_{-r}^{r} \\
&= r\sin^{-1}(1) - r\sin^{-1}(-1) \\
&= 2r\sin^{-1}(1) \\
&= 2r\frac{\pi}{2} \\
&= \pi r.
\end{aligned}$$

Therefore the circumference of a circle of radius r is $2L = 2\pi r$.

Example. Find the arc length of the curve $y = \ln(\sec x)$ on $\left[0, \frac{\pi}{3}\right]$.

Solution. By the chain rule,

$$\frac{dy}{dx} = \frac{1}{\sec x}\frac{d}{dx}(\sec x) = \frac{1}{\sec x}\sec x \tan x = \tan x.$$

The arc length of the curve $y = \ln(\sec x)$ on $\left[0, \frac{\pi}{3}\right]$ is

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
=& \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx \\
=& \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\, dx \\
=& \int_0^{\frac{\pi}{3}} \sec x\, dx \\
=& \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{3}} \\
=& \ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \ln|\sec 0 + \tan 0| \\
=& \ln(2 + \sqrt{3}).
\end{aligned}$$

The roles of x and y in the formula of arc length are switched when the graph is given by $x = f(y)$ from $y = c$ to $y = d$:

$$L = \int_c^d \sqrt{1 + \left[\frac{df}{dy}\right]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Exercises

1. Find the arc length of the curve $y = \dfrac{2x^2 - \ln x}{4}$ over $[1, 2]$, i.e., between $x = 1$ and $x = 2$.
2. Find the arc length of the curve $y = \dfrac{x^6}{12} + \dfrac{1}{8x^4}$ over $\left[\frac{1}{2}, 2\right]$.
3. Find the arc length of $x = y^{\frac{3}{2}}$ between $(0, 0)$ and $(8, 4)$ (Hint. set up in terms of y).

Answers

1. $\dfrac{6+\ln 2}{4}$

2. $\dfrac{1875}{256}$

3. $\dfrac{8(10\sqrt{10}-1)}{27}$

3.7 Work

Suppose f is a continuous function on $[a,b]$. Suppose a particle is moving on the x-axis under a force field that exerts force $f(x)$ when the particle is at x. Then the work done W in moving the particle from $x=a$ to $x=b$ is

$$W = \int_a^b f(x)\,dx.$$

Proof. Break $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ where $x_i = a+i\Delta x$ and $\Delta x = (b-a)/n$. Choose x_i^* from $[x_{i-1},x_i]$, $i=1,2,\ldots,n$. When the particle is on $[x_{i-1},x_i]$, the force acting on the particle is approximately $f(x_i^*)$. Then the work done W_i in moving the particle from x_{i-1} to x_i is approximately $f(x_i^*)\Delta x$. Therefore the total work done W is

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This approximation of W gets better as $n\to\infty$. Thus

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

□

Example. A force of $\sin\left(\frac{\pi x}{4}\right)$ Newtons is acting on a particle moving on a line when it is x meters from the origin. How much work is required in moving the particle from $x=1$ to $x=9$?

Solution. Here the force is $f(x)=\sin\left(\frac{\pi x}{4}\right)$ N when the particle is x m from the origin on the x-axis. Thus required work is

$$\int_1^9 \sin\left(\frac{\pi x}{4}\right)\,dx = -\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\Big|_1^9 = -\frac{4}{\pi}\cos\left(\frac{9\pi}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi}{4}\right) = -\frac{4}{\pi\sqrt{2}}+\frac{4}{\pi\sqrt{2}} = 0.$$

Now we discuss the problems of finding the work required to stretch or compress a spring.

Hooke's Law: The force f required to maintain a spring stretched or compressed x units beyond its natural length is $f(x) = kx$ where $k > 0$ is the spring constant.

Suppose the natural length of a spring is x_0 m and k is the spring constant. Then the work required to stretch it from x_1 m to x_2 m is

$$\int_{x_1-x_0}^{x_2-x_0} f(x)\,dx = \int_{x_1-x_0}^{x_2-x_0} kx\,dx\ J,$$

where $J = Nm$.

Example. A spring has a natural length of 20 cm and 15 N force is required to keep it stretched to a length of 25 cm. How much work is required to stretch it from 20 cm to 30 cm?

Solution. Since 15 N force is required to stretch $0.25 - 0.2 = 0.05$ m, by Hooke's Law,

$$15 = f(0.05) = k \cdot 0.05 \implies k = \frac{15}{0.05} = 300\ \text{N/m}.$$

Then $f(x) = 300x$. The work required to stretch the spring from 0.2 m to 0.3 m is

$$\int_{0.2-0.2}^{0.3-0.2} 300x\,dx = \int_0^{0.1} 300x\,dx = 150x^2\Big|_0^{0.1} = 1.5\ J.$$

Now we discuss the problems of finding the work required to pump liquid out of a tank.

Example. Consider a spherical tank of radius 6 m. Suppose the tank is filled with a liquid with density 1200 kg/m^3 to a height of 4 m. Find the work required to empty the tank through a whole at the top of the tank. Assume $g = 9.8$ m/s^2.

Solution. First we approximate the required work W by approximating the work required to pump a thin layer of liquid to the top of the tank. Draw the origin at the bottom of the sphere and the y-axis vertically up through the center of the sphere.

Since the tank is filled to a height of 4 m, we partition the liquid into n layers by breaking $[0, 4]$ into n subintervals $[y_0, y_1], [y_1, y_2], \ldots, [y_{n-1}, y_n]$ where $y_i = i\Delta y$ and $\Delta y = 4/n$. The ith layer of the liquid is approximately a circular cylinder of radius $\sqrt{6^2 - (6 - y_i)^2} = \sqrt{12y_i - y_i^2}$ and height Δy. So the volume V_i of the ith layer is $V_i \approx \pi(12y_i - y_i^2)\Delta y$ and its mass m_i is

$$m_i = 1200V_i \approx 1200\pi(12y_i - y_i^2)\Delta y.$$

The force f_i on the ith layer to overcome the gravitational force on it is

$$f_i = m_i g \approx 9.8 \cdot 1200\pi(12y_i - y_i^2)\Delta y = 11760\pi(12y_i - y_i^2)\Delta y.$$

Then the work W_i required in pumping the ith layer to the top of the tank is

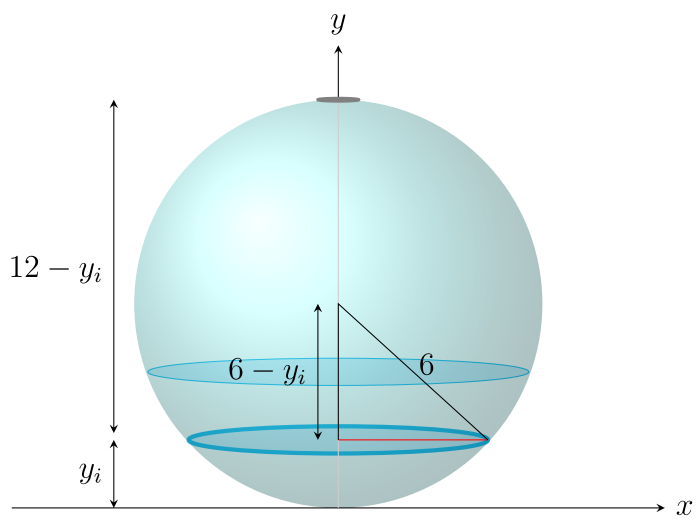
$$W_i \approx f_i(12 - y_i) \approx 11760\pi(12y_i - y_i^2)(12 - y_i)\Delta y = 11760\pi(144y_i - 24y_i^2 + y_i^3)\Delta y.$$

Therefore the total work W required is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 11760\pi(144y_i - 24y_i^2 + y_i^3)\Delta y.$$

This approximation of W gets better as $n \to \infty$. Thus

$$\begin{aligned} W = \lim_{n\to\infty} \sum_{i=1}^{n} 11760\pi(144y_i - 24y_i^2 + y_i^3)\Delta y &= \int_0^4 11760\pi(144y - 24y^2 + y^3)\, dy \\ &= 11760\pi \int_0^4 (144y - 24y^2 + y^3)\, dy \\ &= 11760\pi \left(72y^2 - 8y^3 + \frac{y^4}{4}\right)\Bigg|_0^4 \\ &= 8279040\pi\ J. \end{aligned}$$



Exercises

1. Consider a rectangular tank with height 5 m and base 4×8 m^2. Suppose the tank is half full of water. Find the work required to empty the tank through a whole at the top of the tank. The density of water is 1000 kg/m^3 and $g = 9.8$ m/s^2.

2. Consider a tank with the shape of an inverted circular cone with height 10 m and base radius 4 m. Suppose the tank is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. The density of water is 1000 kg/m^3 and $g = 9.8$ m/s^2.

3. Suppose that 2 J work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch the spring from 35 cm to 40 cm?

Answers

1. 2.94×10^6 J
2. $1568\pi(\frac{2048}{3}) \approx 3.4 \times 10^6$ J
3. $\frac{25}{24} \approx 1.04$ J

Chapter 4

Introduction to Differential Equations

In this chapter we learn the basics of differential equations. This chapter can been treated as applications of integrals. Also, this chapter is written for the students who might not take a class for differential equations.

4.1 Separable Differential Equations

A differential equation is an equation containing a function and its derivatives.

Example.

1. $\dfrac{dy}{dx} = y \sin x$

2. $\dfrac{dv}{dt} = g - \dfrac{\gamma}{m} v$ (motion of a falling object)

3. $\dfrac{d^2\theta}{dt^2} + \dfrac{g \sin\theta}{L} = 0$ (motion of a pendulum)

4. $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$ (one-dimensional heat equation)

A differential equation that does not contain partial derivatives is called an **ordinary differential equations (ODE)**. Note that an ODE has only one independent variable. The **order** of a differential equation is the the order of the highest derivative in it. Differential equations in above examples 1 and 2 have order 1. Differential equation in example 3 has order 2.

A **separable ODE** is of the form

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}.$$

Steps to solve:

$$N(y)\,dy = M(x)\,dx$$
$$\int N(y)\,dy = \int M(x)\,dx + c.$$

Example. Solve the following **IVP** (initial value problem)

$$\frac{dy}{dx} = \frac{2x-3}{y-5},\; y(0)=3 \tag{4.1}$$

and find the valid interval of the solution.

Solution.

$$\frac{dy}{dx} = \frac{2x-3}{y-5}$$
$$(y-5)\,dy = (2x-3)\,dx$$
$$\int (y-5)\,dy = \int (2x-3)\,dx$$
$$\frac{y^2}{2} - 5y = 2\frac{x^2}{2} - 3x + c$$
$$y^2 - 10y = 2(x^2 - 3x + c)$$
$$y^2 - 10y - 2x^2 + 6x - 2c = 0 \tag{4.2}$$

Note that the initial condition is $y(0) = 3$. So we have

$$3^2 - 10 \cdot 3 - 2c = 0$$
$$2c = -21$$

From (4.2) we get

$$y^2 - 10y - 2x^2 + 6x + 21 = 0 \text{ (implicit solution)}$$
$$y = \frac{10 \pm \sqrt{100 - 4(-2x^2 + 6x + 21)}}{2}$$
$$y = \frac{10 \pm 2\sqrt{25 - (-2x^2 + 6x + 21)}}{2}$$
$$y = 5 \pm \sqrt{2x^2 - 6x + 4}$$

So we have two possible solutions:

$$y = 5 + \sqrt{2x^2 - 6x + 4} \text{ and } y = 5 - \sqrt{2x^2 - 6x + 4}.$$

Note that the first one does note satisfy the initial condition $y(0) = 3$. Thus the solution is

$$y = 5 - \sqrt{2x^2 - 6x + 4} \text{ (explicit solution)}.$$

The solution is defined when

$$\begin{aligned} 2x^2 - 6x + 4 &\geq 0 \\ 2(x^2 - 3x + 2) &\geq 0 \\ 2(x-1)(x-2) &\geq 0 \end{aligned}$$

The domain of y is $(-\infty, 1] \cup [2, \infty)$. Because of the initial condition $y(0) = 3$, the valid interval of the solution is $(-\infty, 1]$.

Example. Solve the following ODE.

$$\frac{dy}{dx} = e^{-y} x \cos x$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{x \cos x}{e^y} \\ e^y \, dy &= x \cos x \, dx \\ \int e^y \, dy &= \int x \cos x \, dx \\ e^y &= \int u \, dv && u = x, \; dv = \cos x \, dx \\ &= uv - \int v \, du && du = dx, \; v = \int dv = \int \cos x \, dx = \sin x \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + c \end{aligned}$$

The general solution is $y = \ln |x \sin x + \cos x + c|$.

Exercises

1. Determine if $y = e^x \cos x$ is a solution of the ODE $y' - y = e^x \sin x$.
2. Find the order of the ODEs: (a) $y''' + \sin(t + y) = t^3$, (b) $e^y y'' + y' = e^t$.
3. Solve the following ODEs.
   (a) $\dfrac{dy}{dx} = \dfrac{e^{-y^2}}{2y(1+x^2)}, \; y(1) = 0$
   (b) $y' = \dfrac{y^2 - y}{x}$
   (c) $t^2 \dfrac{dy}{dt} = \ln t$

Answers

1. Not a solution
2. (a) 3, (b) 2
3. (a) $y^2 = \ln|\tan^{-1} x + (1 - \frac{\pi}{4})|$

   (b) $y = \dfrac{1}{1 - cx}$

   (c) $y = \dfrac{1 + \ln t}{-t} + c$

4.2 Modeling with First Order ODEs

Radioactive Decay: Let $N(t)$ be the mass of a radioactive element at time t. The rate of change (decay) of N is proportional to its current value.

$$\frac{dN}{dt} = -kN,$$

where $k > 0$ depends on the element. Solving this ODE we get $N = ce^{-kt}$. Suppose the initial amount is $N_0 = N(0)$. Then we get $c = N_0$. Thus the solution is

$$N(t) = N_0 e^{-kt}.$$

Carbon Dating: There are 2 types of carbon atoms: ^{12}C (the stable nuclide) and radioactive ^{14}C with a halflife of about 5,730 years. The ratio $^{14}C :^{12} C$ is approximately constant in nature. A living creature taking carbon from nature is made up with this ratio. After its death ^{14}C starts to decay.

Example. Suppose 25% of the original amount of ^{14}C remained in a fossil. Find the age of the fossil.

Solution. First we find k for ^{14}C in $N(t) = N_0 e^{-kt}$. We know that $\frac{N_0}{2} = N_0 e^{-5730k}$. Solving we get $k = \ln 2/5730$ and hence

$$N(t) = N_0 e^{-t \ln 2/5730}.$$

For the fossil we want to find t for which $N(t) = 25N_0/100$. Plugging this into the preceding equation we get

$$25N_0/100 = N_0 e^{-t \ln 2/5730} \implies t = 11,460 \text{ years}$$

Newton's Law of Cooling: Let $T(t)$ be the temperature of an object at time t. Then

$$\frac{dT}{dt} = -k(T - T_a),$$

where $k > 0$ is a constant and T_a is the constant ambient temperature. Solving this ODE we get $T = T_a + ce^{-kt}$. Suppose the initial temperature is $T_0 = T(0)$. Then we get $c = T_0 - T_a$. Thus the solution is

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

Note that $T(t) = T_a + (T_0 - T_a)e^{-kt} \to T_a$ as $t \to \infty$.

Kirchhoff's Circuit Law: Consider an electric circuit with a capacitor, resistor, and battery. Let $Q(t)$ be the charge of the capacitor at time t. Then

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the constant resistant, C is the constant capacitance, and V is the constant voltage supplied. Solving this ODE we get $Q = CV + ke^{-t/RC}$. Suppose the initial charge is $Q(0) = 0$. Then we get $k = -CV$. Thus the solution is

$$Q(t) = CV(1 - e^{-t/RC}).$$

Note that $Q(t) = CV(1 - e^{-t/RC}) \to CV$ as $t \to \infty$.

Exercises

1. Suppose 10% of the original amount of ^{14}C remained in a fossil. Set up and solve an ODE to find the age of the fossil. Assume that ^{14}C has halflife of 5,730 years.

2. Suppose you have a bowl of soup at 100° C. After 6 minutes it cools down to 90° C by occasional stirring. Set up and solve an ODE to find the time after which the soup would be 50° C. Assume the room temperature to be 20° C.

Answers

1. 19034.6 years

2. $6\ln(3/8)/\ln(7/8) \approx 44.07$ min

4.3 Euler's Method

Can we numerically find an approximate solution to the following ODE?

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

Euler's Method: The tangent line to a solution $y = \phi(x)$ at the point (x_0, y_0) has slope $\left.\frac{dy}{dx}\right]_{(x_0,y_0)} = f(x_0, y_0)$. So an equation of the tangent line would be

$$y = y_0 + (x - x_0)f(x_0, y_0).$$

If x_1 is close to x_0, then $y_1 = y_0 + (x_1 - x_0)f(x_0, y_0)$ would be a good approximation to $y = \phi(x_1)$. Similarly if x_2 is close to x_1, then $y_2 = y_1 + (x_2 - x_1)f(x_1, y_1)$ would be a good approximation to $y = \phi(x_2)$. Thus a general formula for this tangent line expression would be

$$y_{n+1} = y_n + (x_{n+1} - x_n)f(x_n, y_n),\ n = 0, 1, 2, \ldots. \tag{4.3}$$

If we take $x_0, x_1, x_3, \ldots$ with a fixed difference h (i.e. $x_{n+1} = x_n + h$), then (4.3) becomes

$$y_{n+1} = y_n + hf(x_n, y_n),\ n = 0, 1, 2, \ldots. \tag{4.4}$$

Example. Use Euler's method with step size $h = 0.5$ to approximate $y(4)$ where $y = \phi(x)$ is a solution to the following ODE.

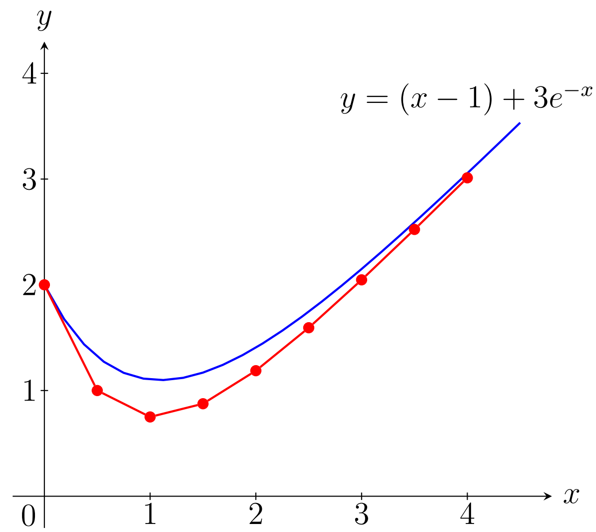
$$\frac{dy}{dx} = x - y,\ y(0) = 2$$

Solution. We have $h = 0.5$, $x_0 = 0$, $y_0 = 2$ and $f(x, y) = x - y$. So from (4.4) we have $y_{n+1} = y_n + 0.5(x_n - y_n)$, $n = 0, 1, 2, \ldots$.

$$x_1 = x_0 + 0.5 = 0.5, \qquad y_1 = y_0 + 0.5(x_0 - y_0) = 2 + 0.5(0 - 2) = 1$$
$$x_2 = x_1 + 0.5 = 1, \qquad y_2 = y_1 + 0.5(x_1 - y_1) = 1 + 0.5(0.5 - 1) = 0.75 \textit{ etc.}$$

n	x_n	y_n
0	0	2
1	0.5	1
2	1	0.75
3	1.5	0.875
4	2	1.1875
5	2.5	1.59375
6	3	2.046875
7	3.5	2.5234375
8	4	3.01171875

So $y(4) \approx y_8 = 3.01171875$, i.e., $y(4)$ is approximately $y_8 = 3.01171875$.



Euler's tangent line approximation to $\frac{dy}{dx} = x - y$, $y(0) = 2$ with $h = 0.5$

Checking solution: Solving $\dfrac{dy}{dx} = x - y$, $y(0) = 2$, we get $y = (x-1) + 3e^{-x}$. The exact value of $y(4)$ is $(4-1) + 3e^{-4} = 3.0183$ which was approximated by 3.0117. (pretty close!)

Exercises

1. Use Euler's method with step size $h = 0.25$ to approximate the solution of the IVP:

$$\frac{dy}{dt} = (1-y)\cos t,\ y(0) = 3$$

2. Use Euler's method with step size $h = 0.5$ to approximate the solution of the IVP:

$$\frac{dy}{dt} = \frac{e^{y^2}}{t},\ 1 \le t \le 3,\ y(1) = 0$$

Answers

1. $y(2) \approx 1.5923$
2.

i	t_i	y_i
0	1	0
1	1.5	0.5
2	2	$fill$
3	2.5	$fill$
4	3	3.5315

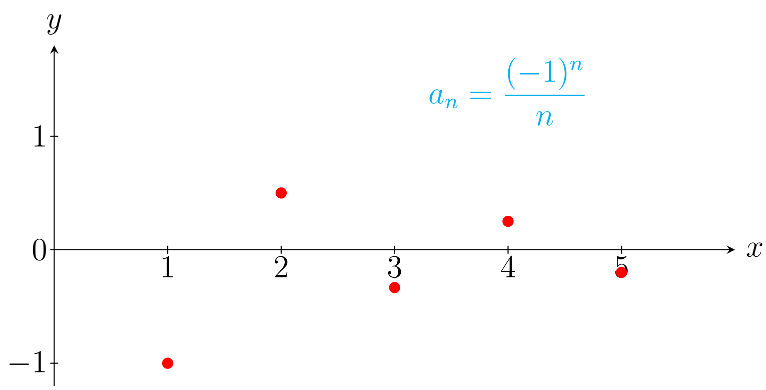
Chapter 5

Infinite Sequences and Series

In this chapter we learn the basics of sequence and series. The main goal of this chapter is to write a function as a power series and then integrate it by integrating the power series term by term.

5.1 Sequences

A *sequence* $\{a_n\}$ of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$. Sometimes it is denoted by (a_n), $\{a_n\}_{n=1}^{\infty}$ or by a few terms with a certain pattern $\{a_1, a_2, a_3, \ldots\}$. A sequence can also be defined recursively.



Example.

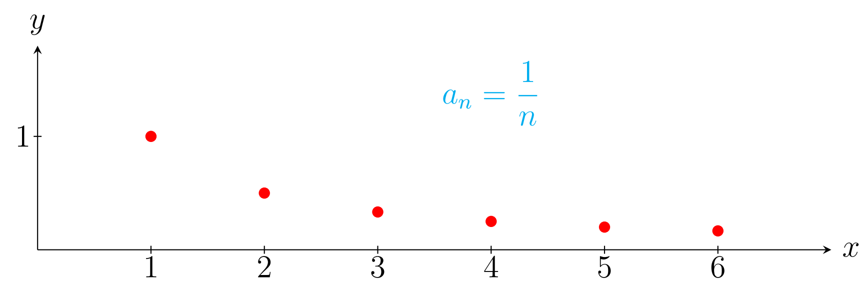
1. First four terms of the sequence $\left\{\frac{(-1)^n}{n}\right\}$ are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}$.

2. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ can be written as $\left\{\frac{1}{n}\right\}$.

3. The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1.$$

Definition (Convergence)**.** A sequence $\{a_n\}$ *converges* to a real number L if x_n can be made arbitrarily close to L by taking sufficiently large n (for details, see the $\varepsilon - N$ definition). It is denoted by $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$. The sequence $\{a_n\}$ *diverges* if it is not convergent.

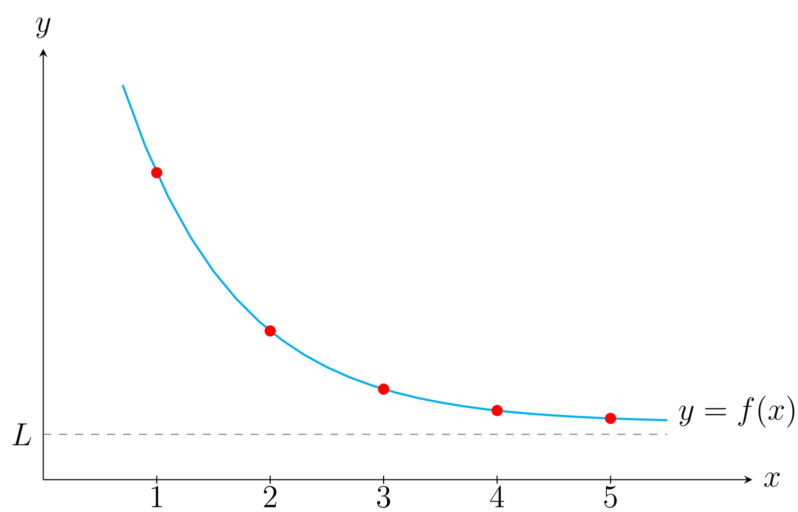


Example.

1. The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.
2. $\{n^2\}$ diverges to ∞.
3. $\{(-1)^n\}$ diverges as its terms oscillate between 1 and -1.

Note that if a sequence $\{x_n\}$ diverges, its terms oscillate indefinitely like that of $\{(-1)^n\}$ or it diverges to $\pm\infty$.

Theorem 5.1.1. *If $a_n = f(n)$ for all integers $n \geq 1$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.*



Example. Show that $\lim_{n\to\infty} \dfrac{n + \tan^{-1} n}{n} = 1$.

Solution. Let $f(x) = \dfrac{x + \tan^{-1} x}{x}$. Note that $f(n)$ is the nth term of the sequence $\left\{\dfrac{n + \tan^{-1} n}{n}\right\}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x + \tan^{-1} x}{x} = \lim_{x\to\infty} \frac{1 + \frac{1}{1+x^2}}{1} = 1 \quad \text{(by l'Hospitals's Rule)}.$$

Thus $\displaystyle\lim_{n\to\infty} \frac{n + \tan^{-1} n}{n} = 1$, i.e., the sequence $\left\{\dfrac{n + \tan^{-1} n}{n}\right\}$ converges to 1.

Algebra of limits for sequences:

Theorem 5.1.2. *The following are true for any convergent sequences* $\{a_n\}$ *and* $\{b_n\}$*:*

(a) $\displaystyle\lim_{n\to\infty} (a_n \pm b_n) = \left(\lim_{n\to\infty} a_n\right) \pm \left(\lim_{n\to\infty} b_n\right)$.

(b) $\displaystyle\lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n$ *for all real numbers* c.

(c) $\displaystyle\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$.

(d) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *whenever* $\displaystyle\lim_{n\to\infty} b_n \neq 0$.

(e) $\displaystyle\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ *for all real numbers* $p > 0$ *whenever* $a_n > 0$.

Note that the convergence in the assumption is crucial. For $\{a_n\} = \{\frac{1}{n}\}$ and $\{b_n\} = \{n\}$, we have $\{a_n b_n\} = \{1\}$. But

$$\lim_{n\to\infty} a_n b_n = 1 \neq 0 \cdot \infty = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

Example. Determine if the sequence $\{a_n\}$ converges or diverges.
(a) $a_n = \dfrac{n^3}{n^3+1}$, (b) $a_n = \dfrac{n^3}{n+1}$, (c) $a_n = \sqrt{\dfrac{n+1}{9n+1}}$.

Solution. (a) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = \frac{1}{1 + \lim_{n\to\infty} \frac{1}{n^3}} = \frac{1}{1+0} = 1$. Thus $\left\{\dfrac{n^3}{n^3+1}\right\}$ converges to 1.

(b) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{n+1} = \lim_{n\to\infty} \frac{n^2}{1+\frac{1}{n}} = \infty$. Thus $\left\{\dfrac{n^3}{n+1}\right\}$ diverges to ∞.

(c) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{1+\frac{1}{n}}{9+\frac{1}{n}}} = \sqrt{\frac{1+0}{9+0}} = \frac{1}{3}$. Thus $\left\{\sqrt{\dfrac{n+1}{9n+1}}\right\}$ converges to $\frac{1}{3}$.

Theorem 5.1.3. *If $\{a_n\}$ is convergent and f is a continuous function, then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

Example. Since $f(x) = \cos x$ is a continuous function,

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{2}{n}\right) = \cos 0 = 1.$$

Thus $\left\{\cos\left(\frac{2}{n}\right)\right\}$ converges to 1.

Theorem 5.1.4 (Sandwich Theorem)**.** *Suppose $a_n \le b_n \le c_n$ for all natural numbers n. If both $\{a_n\}$ and $\{c_n\}$ converge to L, then $\{b_n\}$ also converges to L.*

Example. Show that $\displaystyle\lim_{n\to\infty} \frac{\sin(3n)}{n} = 0$.

Solution. First note that $-1 \le \sin(3n) \le 1$ for all real numbers n. Then for all natural numbers n,

$$-\frac{1}{n} \le \frac{\sin(3n)}{n} \le \frac{1}{n}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, we have $\displaystyle\lim_{n\to\infty} \frac{\sin(3n)}{n} = 0$ by the Sandwich Theorem.

As consequences of the Sandwich Theorem, we have the following results:

(a) If $\displaystyle\lim_{n\to\infty} |a_n| = 0$, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

(b) $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent otherwise. In particular,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

Proof. (a) $-|a_n| \le a_n \le |a_n|$ for all natural numbers n. Now apply the Sandwich Theorem.
(b) If $r = 1$, then $\{r^n\} = \{1\}$ is convergent. Suppose $-1 < r < 1$. Then $0 \le |r| < 1$. Since $\displaystyle\lim_{x\to\infty} |r|^x = 0$, $\displaystyle\lim_{n\to\infty} |r|^n = 0$. Since $|r|^n = |r^n|$, $\displaystyle\lim_{n\to\infty} r^n = 0$ by (a).

For $r > 1$, $\displaystyle\lim_{x\to\infty} r^x = \infty$ and hence $\displaystyle\lim_{n\to\infty} r^n = \infty$. For $r = -1$, note that $\{(-1)^n\}$ is divergent as its terms oscillate between 1 and -1. Similar arguments can be applied for $r < -1$. □

Example. (a) Since $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$, $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

(b) $\displaystyle\lim_{n\to\infty} \frac{2^n(-1)^n}{3^n} = \lim_{n\to\infty} \left(-\frac{2}{3}\right)^n = 0$ as $-1 < -\frac{2}{3} \le 1$. But $\displaystyle\lim_{n\to\infty} \left(\frac{3}{2}\right)^n$ does not exist as $\frac{3}{2} > 1$.

Definition (Bounded Sequence)**.** A sequence $\{a_n\}$ is *bounded* if there exists a real number M such that $|a_n| \leq M$ for all natural numbers n.

Example. $\{\frac{1}{n}\}$ is bounded since $|\frac{1}{n}| \leq 1$ for all natural numbers n.

Theorem 5.1.5. *A convergent sequence is bounded.*

Definition (Increasing/decreasing sequence)**.** A sequence $\{a_n\}$ is *increasing* if $a_{n+1} > a_n$ for all natural numbers n and *decreasing* if $a_{n+1} < a_n$ for all natural numbers n.

Theorem 5.1.6 (Monotone Convergence Theorem)**.**

(a) An increasing sequence that is bounded above is convergent.

(b) A decreasing sequence that is bounded below is convergent.

Note that the Monotone Convergence Theorem is also true for non-decreasing and non-increasing sequences respectively.

Example.

1. $\{\frac{1}{n}\}$ is a decreasing sequence as $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all natural numbers n. Also $\{\frac{1}{n}\}$ is bounded below by 0 as $0 < \frac{1}{n} = a_n$ for all natural numbers n. Since $\{\frac{1}{n}\}$ is decreasing and bounded below, it is convergent by the Monotone Convergence Theorem.

2. $\{x_n\}$ where $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}},\ n \geq 2$.
   We show $\{x_n\}$ is increasing by induction. First note that

   $$x_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = \sqrt{2 + x_1} = x_2.$$

   Assume $x_{n-1} < x_n$ for some $n \in \mathbb{N}$. We show $x_n < x_{n+1}$. Note that

   $$\begin{aligned} x_{n+1}^2 - x_n^2 &= (2 + x_n) - (2 + x_{n-1}) = x_n - x_{n-1} \\ \implies x_{n+1} - x_n &= (x_n - x_{n-1})/(x_{n+1} + x_n) \end{aligned}$$

   Since $x_{n-1} < x_n$ and $x_n, x_{n+1} > 0$, we have $x_{n+1} - x_n = (x_n - x_{n-1})/(x_{n+1} + x_n) > 0$, i.e., $x_n < x_{n+1}$.

   We show $\{x_n\}$ is bounded above also by induction. Note that $x_1 = \sqrt{2} < 3$ and $x_2 = \sqrt{2 + \sqrt{2}} < 3$. Assume $x_n < 3$ for some $n \in \mathbb{N}$. We show $x_{n+1} < 3$. Note that

   $$x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 3} < 3.$$

   By mathematical induction $x_n < 3$ for all natural numbers n.

   Since $\{x_n\}$ is an increasing sequence bounded above by 3, it is convergent by the MCT. Suppose $\lim\limits_{n\to\infty} x_n = L$. To find L, note that $x_n^2 = 2 + x_{n-1},\ n \geq 2$. Taking limit of both sides we get

   $$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty}(2 + x_{n-1}) \implies (\lim_{n\to\infty} x_n)^2 = 2 + \lim_{n\to\infty} x_{n-1} \implies L^2 = 2 + L \implies L = -1, 2.$$

   Since $x_n > 0$ for all natural numbers n, $L \geq 0$. Thus $L = 2$, i.e., $\lim\limits_{n\to\infty} x_n = 2$.

Exercises

1. Determine whether the following sequences are convergent or divergent. If one is convergent, find its limit. Justify your answers.
(a) $\left\{\dfrac{e^n - e^{-n}}{e^{2n}+1}\right\}$, (b) $\left\{\dfrac{n^2}{\sqrt{n^3+2}}\right\}$, (c) $\{\ln(2+(0.5)^n)\}$, (d) $\left\{\dfrac{\cos n}{n}\right\}$, (e) $\{(-1)^n n e^{-n}\}$.

2. Let $a_n = \dfrac{n-1}{n}$. Determine if the sequence $\{a_n\}$ is increasing, decreasing, bounded. Determine if $\{a_n\}$ converges by the Monotone Convergence Theorem.

Answers

1. (a) limit$=0$, (b) divergent, (c) limit$=\ln 2$, (d) limit$=0$, (e) limit$=0$

2. Increasing, bounded above by 1, convergent

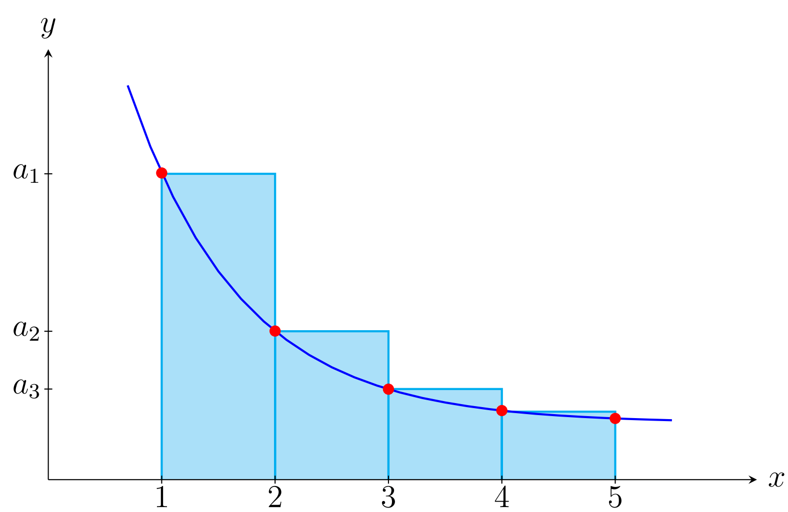
5.2 Series

A *series* is the sum of the terms of a sequence $\{a_n\}$ which is denoted by $\sum_{n=1}^{\infty} a_n$.

Example.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ (a geometric series).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$ (a telescoping series).



A series $\sum_{n=1}^{\infty} a_n$ as an area

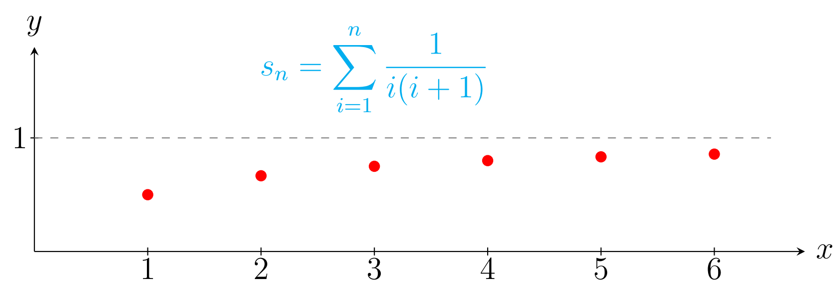
Definition. For the series $\sum_{n=1}^{\infty} a_n$, the *nth partial sum*, denoted by s_n, is

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

The series $\sum_{n=1}^{\infty} a_n$ *converges* to a real number s if the sequence $\{s_n\}$ of partial sums of $\sum_{n=1}^{\infty} a_n$ converges to s. We write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

A series is called *convergent* if it converges. Otherwise it is *divergent*.



Example. Consider the telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$.

Note that $s_1 = \frac{1}{1\cdot 2}$, $s_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$, $s_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4}, \ldots$.

$$\begin{aligned}
s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} \\
&= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1}.
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Theorem. *The geometric series* $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ *is convergent if* $|r| < 1$, *i.e.,* $-1 < r < 1$ *and divergent if* $|r| \geq 1$. *In particular,*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

Proof.

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \implies rs_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Then

$$s_n - rs_n = a - ar^n \implies (1-r)s_n = a(1-r^n) \implies s_n = a\frac{1-r^n}{1-r}$$

Since $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}.$$

□

Example.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ where the first term $a = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$ is in $(-1, 1)$.

(b) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1} = \frac{e}{1-\frac{e}{3}} = \frac{3e}{3-e}$ where the first term $a = e$ and the common ratio $r = \frac{e}{3}$ is in $(-1, 1)$.

(c) We show that $0.\bar{3} = \frac{1}{3}$.

$$0.\bar{3} = 0.333\cdots = 0.3 + 0.03 + 0.003 + \cdots = \sum_{n=1}^{\infty} \frac{3}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{1}{3},$$

where $a = \frac{3}{10}$ and $r = \frac{1}{10}$ is in $(-1, 1)$.

(d) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ whenever $|x| < 1$. Note that $\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1}$ where $a = 1$ and $r = x$ is in $(-1, 1)$.

Algebra for series:

Theorem. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent, then so are the following series:*

$$\sum_{n=1}^{\infty}(a_n + b_n),\ \sum_{n=1}^{\infty}(a_n - b_n),\ \sum_{n=1}^{\infty} ca_n,$$

for all real numbers c. *In particular,*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,\ \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n,\ \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

Example. Find $\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)} + \frac{1}{e^n}\right)$.

Solution. We know that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$ (the telescoping series) and $\sum_{n=1}^{\infty}\frac{1}{e^n} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$ (the geometric series with $a = r = \frac{1}{e}$). Then

$$\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)} + \frac{1}{e^n}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{e^n} = 2\cdot 1 + \frac{1}{e-1} = \frac{2e-1}{e-1}.$$

Theorem. *If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = s$. Since $a_n = s_n - s_{n-1}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

□

Note that the converse of the preceding theorem is not true.

Example. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent but $\lim_{n\to\infty}\frac{1}{n} = 0$.

Proof.

$$\begin{aligned}
s_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \\
&= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right) \\
&\geq 1 + 2^0\left(\frac{1}{2}\right) + 2^1\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{8}\right) \cdots + 2^{n-1}\left(\frac{1}{2^n}\right) \\
&= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots + \left(\frac{1}{2}\right) \\
&= 1 + n\left(\frac{1}{2}\right)
\end{aligned}$$

Thus $s_{2^n} \geq 1 + \frac{n}{2}$ for all natural numbers n. Then $\lim\limits_{n\to\infty} s_{2^n} = \infty$ which implies

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} s_n = \infty.$$

□

By the contrapositive of the preceding theorem, we have the following result:

Theorem (Divergence Test)**.** *If* $\lim\limits_{n\to\infty} a_n \neq 0$ *or the limit does not exist, the* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent.*

Example. $\sum\limits_{n=1}^{\infty} \frac{n(n+3)}{(n+1)^2}$ is divergent by the Divergence Test because

$$\lim_{n\to\infty} \frac{n(n+3)}{(n+1)^2} = \lim_{n\to\infty} \frac{1\left(1+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)^2} = 1 \neq 0.$$

Exercises

1. Consider the telescoping series $\sum\limits_{n=1}^{\infty} \frac{2}{n(n+2)}$.
   (a) Find s_n, the nth partial sum.
   (b) Use s_n to find $\sum\limits_{n=1}^{\infty} \frac{2}{n(n+2)}$.
2. Write $0.\overline{12}$ as a geometric series to find the sum.

3. Determine whether each of the following series is convergent or divergent. If it is convergent, find its sum.
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$,(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{4^{n+1}}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$.

Answers

1. (a) $s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, (b) $\frac{3}{2}$
2. $\frac{12}{99}$
3. (a) $\frac{5}{2}$, (b) Divergent, (c) $\frac{1}{4-\pi}$, (d) Divergent

5.3 Series with Positive Terms

In this section we study four tests of convergence for series with positive terms: The Integral Test, the p-series Test, the Comparison Test, the Limit Comparison Test.

The Integral Test: Let $a_n = f(n)$ for all integers $n \geq 1$ where f is a continuous, positive, and decreasing function on $[1, \infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent.

Proof. Since f is decreasing,

$$0 < a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx \implies a_1 < s_n \leq a_1 + \int_1^n f(x)\,dx.$$

Taking limit as $n \to \infty$,

$$a_1 \leq \lim_{n\to\infty} s_n \leq a_1 + \lim_{n\to\infty} \int_1^n f(x)\,dx \implies a_1 \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx.$$

If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then

$$a_1 \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx < \infty.$$

Similarly

$$\int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_{n-1} = s_{n-1}$$
$$\implies \int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx \leq \lim_{n\to\infty} s_{n-1} = \sum_{n=1}^{\infty} a_n.$$

If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then

$$\infty = \int_1^\infty f(x)\,dx \leq \sum_{n=1}^\infty a_n \implies \sum_{n=1}^\infty a_n = \infty.$$

□

Example. Use the Integral Test to show that $\displaystyle\sum_{n=1}^\infty \frac{2n}{n^4+3}$ is convergent.

Solution. First note that $f(x) = \dfrac{2x}{x^4+3}$ is a continuous, positive, and decreasing function on $[1,\infty)$.

$$\begin{aligned}
\int_1^\infty f(x)\,dx &= \int_1^\infty \frac{2x}{x^4+3}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{2x}{x^4+3}\,dx \\
&= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right)\Bigg|_1^t \quad (\text{Hint. } u = x^2) \\
&= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \left[\tan^{-1}\left(\frac{t^2}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] \\
&= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] \\
&= \frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

Since $\displaystyle\int_1^\infty \frac{2x}{x^4+3}\,dx$ is convergent, $\displaystyle\sum_{n=1}^\infty \frac{2n}{n^4+3}$ is convergent by the Integral Test.

The p-series Test: The p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Proof. Let $p > 1$. Then $f(x) = \frac{1}{x^p}$ is a continuous, positive, and decreasing function on $[1,\infty)$. Also $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent by the integral p-test. Then by the Integral Test, $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent.

By similar arguments we can prove that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is divergent when $0 < p \leq 1$. Finally for $p \leq 0$, $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is divergent by the Divergence Test. □

Example.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series Test with $p = 2 > 1$.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by the p-series Test with $p = \frac{1}{2} \leq 1$.

The Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms.

(a) If $a_n \leq b_n$ for all integers $n \geq 1$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(b) If $a_n \leq b_n$ for all integers $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

Note that we often compare a series with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ or the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$.

Example.

(a) Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n^5+1}}$ is convergent.

Solution. For all integers $n \geq 1$,

$$0 < \frac{1}{\sqrt[3]{2n^5+1}} \leq \frac{1}{\sqrt[3]{2n^5}} = \frac{1}{\sqrt[3]{2}n^{\frac{5}{3}}}.$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{3}}}$ is convergent by the p-series Test with $p = \frac{5}{3} > 1$. Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2}n^{\frac{5}{3}}} = \frac{1}{\sqrt[3]{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{3}}}$ is convergent. Finally by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n^5+1}}$ is convergent.

(b) Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$ is divergent.

Solution. For all integers $n \geq 1$,

$$\frac{3^n}{2^n - 1} \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n > 0.$$

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ is divergent as a geometric series with the common ratio $r = \frac{3}{2} \geq 1$. Then by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$ is divergent.

The Limit Comparison Test: Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are two series with positive terms. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}$ is a positive number, then either both series converge or both series diverge.

Example. Use the Limit Comparison Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4-1}}{2n^3+n^2}$ is divergent.

Solution. First we construct a series with positive terms by keeping the highest degree terms of the numerator and denominator of the given series:

$$\sum_{n=1}^{\infty}\frac{\sqrt{n^4}}{n^3} = \sum_{n=1}^{\infty}\frac{n^2}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n}.$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{\frac{\sqrt{n^4-1}}{2n^3+n^2}}{\frac{1}{n}} &= \lim_{n\to\infty}\frac{\sqrt{n^4-1}}{2n^3+n^2}\cdot\frac{n}{1} \\
&= \lim_{n\to\infty}\frac{n\sqrt{n^4-1}/n^3}{(2n^3+n^2)/n^3} \\
&= \lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^4}}}{2+\frac{1}{n}} \\
&= \frac{1}{2} > 0.
\end{aligned}$$

Since the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^4-1}}{2n^3+n^2}$ is divergent by the Limit Comparison Test.

Exercises

1. Use (a) the Integral Test, (b) the Comparison Test, and (c) the limit Comparison Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+4}$ is convergent.

2. Use (a) the Integral Test, (b) the Comparison Test, and (c) the limit Comparison Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{n}{4n^2+1}$ is divergent.

3. Determine whether each of the following series is convergent or divergent.
   (a) $\sum_{n=1}^{\infty} ne^{-n}$,(b) $\sum_{n=1}^{\infty} \frac{n^2-1}{5n^4+2}$, (c) $\sum_{n=0}^{\infty} \frac{\cos n}{3^n}$, (d) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

Answers

1. Compare with a bigger convergent p-series
2. Compare with a smaller divergent p-series
3. (a) Convergent by the Integral Test, (b) Convergent by the Comparison Test with a p-series, (c) Convergent by the Comparison Test with a geometric series, (d) Divergent by the Limit Comparison Test with a p-series

5.4 Absolute and Conditional Convergence

An *alternating series* is a series of the form

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots \text{ or } \sum_{n=1}^{\infty}(-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots ,$$

where $\{a_n\}$ is a sequence of positive real numbers.

Example. The following alternating series is called the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Alternating Series Test: The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ are convergent if $\{a_n\}$ is a positive and decreasing sequence that converges to 0, i.e.,

(a) $a_n > 0$ for all integers $n \geq 1$,

(b) $a_{n+1} < a_n$ for all integers $n \geq 1$, and

(c) $\lim_{n\to\infty} a_n = 0$.

Proof. Consider the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ where $\{a_n\}$ is a positive and decreasing sequence that converges to 0. Since $\{a_n\}$ is positive and decreasing,

$$s_{2n} = s_{2n-2} + (-a_{2n-1} + a_{2n}) < s_{2n-2}.$$

Thus $\{s_{2n}\}$ is a decreasing sequence. Note that

$$s_{2n} = -a_1 + (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n}.$$

Since $(a_2 - a_3), \ldots, (a_{2n-2} - a_{2n-1}), a_{2n}$ are positive, $s_{2n} > -a_1$. Thus $\{s_{2n}\}$ is a decreasing sequence that is bounded below. Then by the Monotone Convergence Theorem, $\{s_{2n}\}$ is convergent. Suppose $\lim\limits_{n\to\infty} s_{2n} = s$. Since $s_{2n+1} = s_{2n} - a_{2n+1}$,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} a_{2n+1} = s - 0 = s.$$

Since both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ converge to s, so does $\{s_n\}$. Therefore

$$\sum_{n=1}^{\infty}(-1)^n a_n = \lim_{n\to\infty} s_n = s.$$

□

Example. Use the Alternating Series Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent.

Solution. The given series is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n = \frac{1}{n}$. $\{a_n\} = \{\frac{1}{n}\}$ is a positive and decreasing sequence that converges to 0 because

(a) $a_n = \frac{1}{n} > 0$ for all integers $n \geq 1$,

(b) $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all integers $n \geq 1$, and

(c) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

Thus by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent.

Note:

1. To show that $\{a_n\}$ is decreasing where the formula of $a_n = f(n)$ is more complicated, show that $f'(x) < 0$ on $(1, \infty)$. For the above problem, $f(x) = \frac{1}{x}$ and $f'(x) = -\frac{1}{x^2} < 0$ on $(1, \infty)$. Then $\{a_n\} = \{\frac{1}{n}\}$ is decreasing.

2. In the above example, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent but $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Definition. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if it is convergent but $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is divergent (i.e., a conditionally convergent series is a convergent series that is not absolutely convergent).

Example. Recall that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent but not absolutely convergent.

Theorem. *An absolutely convergent series is convergent, i.e., if* $\sum_{n=1}^{\infty} |a_n|$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

Proof. Suppose that $\sum_{n=1}^{\infty} |a_n|$ is convergent. For all integers $n \geq 1$, $-|a_n| \leq a_n \leq |a_n|$. Adding $|a_n|$, we get

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, so is $\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|$. By the Comparison Test, $\sum_{n=1}^{\infty}(a_n + |a_n|)$ is convergent. Finally $\sum_{n=1}^{\infty} a_n$ is also convergent as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

□

Example. Consider the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ which has positive and negative terms. For all integers $n \geq 1$,

$$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series Test with $p = 2 > 1$, $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ is convergent by the Comparison Test. Since $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ is convergent, $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent and consequently convergent.

To see the importance of the concept of absolute and conditional convergence of a series, we discuss rearrangements of the terms of a series. Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ whose sum is obtained by evaluating the Taylor series of $\ln(1 + x)$ at $x = 1$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Now we rearrange the terms as follows.

$$\begin{aligned}
&\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\cdots\\
=&\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots\\
=&\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)\\
=&\frac{1}{2}\ln 2.
\end{aligned}$$

We can even rearrange the terms to make it converge to any real number. This is due to the conditional convergence of the alternating harmonic series.

Riemann's Rearrangement Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series and L be a real number. Then there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to L.

The above strange phenomenon does not happen for absolutely convergent series.

Theorem. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series converging to L. Then any rearrangement of $\sum_{n=1}^{\infty} a_n$ also converges to L.*

Exercises

1. Determine if each of the following series is absolutely or conditionally convergent:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n^3}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^2}$, (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$.

2. Use the Alternating Series Test to determine the convergence of the following series:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\ln(n+1)}{n}$.

Answers

1. (a) Absolutely convergent, (b) Conditionally convergent, (c) Absolutely convergent, (d) Conditionally convergent

2. Show three conditions of the Alternating Series Test are satisfied

5.5 Ratio and Root Tests

In this section we learn two tests for convergence of series: the Root Test and the Ratio Test.

Ratio Test: Consider a series $\sum_{n=1}^{\infty} a_n$. Let

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(a) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $L = 1$, then this test is inconclusive.

Proof. (a) Suppose $L < 1$. Let r be a number such that $L < r < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < r$, there is a positive integer K such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r, \text{ i.e., } |a_{n+1}| < r|a_n| \text{ for all } n \geq K.$$

Successively applying the last inequality, we get

$$|a_{K+n}| < r^n |a_K| \text{ for all } n \geq 1.$$

Since $\sum_{n=1}^{\infty} r^n |a_K|$ is convergent as a geometric series, $\sum_{n=1}^{\infty} |a_{K+n}|$ is also convergent by the Comparison Test. Then so is the following as the sum of a finite number and a convergent series:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{K} |a_n| + \sum_{n=1}^{\infty} |a_{K+n}|.$$

Thus $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Then there is a positive integer K such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1, \text{ i.e., } |a_{n+1}| > |a_n| \text{ for all } n \geq K.$$

Then $\lim_{n\to\infty} a_n \neq 0$. Thus $\sum_{n=1}^{\infty} a_n$ is divergent by the Divergence Test.

(c) When $L = 1$, $\sum_{n=1}^{\infty} a_n$ may be divergent $\left(\text{e.g., } \sum_{n=1}^{\infty} \frac{1}{n}\right)$, absolutely convergent $\left(\text{e.g., } \sum_{n=1}^{\infty} \frac{1}{n^2}\right)$, and conditionally convergent $\left(\text{e.g., } \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\right)$. □

Example.

1. Use the Ratio Test to determine if $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent or divergent.

   Solution. Here $a_n = \frac{n!}{n^n}$. Then $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$ and

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{(n+1)\cdot(n+1)^n} \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \\
&= \frac{1}{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n} \\
&= \frac{1}{e}.
\end{aligned}
$$

   Since $L = \frac{1}{e} < 1$, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent by the Ratio Test and hence convergent.

2. Use the Ratio Test to determine if $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is absolutely convergent or divergent.

Solution. Here $a_n = \dfrac{2^n}{n^2}$. Then $a_{n+1} = \dfrac{2^{n+1}}{(n+1)^2}$ and

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} \right| \\
&= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \frac{n^2}{2^n} \\
&= \lim_{n\to\infty} \frac{2 \cdot 2^n}{2^n} \left(\frac{n}{n+1} \right)^2 \\
&= \lim_{n\to\infty} 2 \left(\frac{1}{1+\frac{1}{n}} \right)^2 \\
&= 2 \left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \right)^2 \\
&= 2.
\end{aligned}$$

Since $L = 2 > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is divergent by the Ratio Test.

3. Use the Ratio Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent or divergent.

Solution. Here $a_n = \dfrac{1}{n^2}$. Then $a_{n+1} = \dfrac{1}{(n+1)^2}$ and

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \frac{1}{(n+1)^2} \frac{n^2}{1} \\
&= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^2 \\
&= \left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \right)^2 \\
&= 1.
\end{aligned}$$

Since $L = 1$, the Ratio Test is inconclusive (although $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series Test). Similarly we can show for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ that $L = 1$ and the Ratio Test is inconclusive where the series is known to be divergent.

Root Test: Consider a series $\sum_{n=1}^{\infty} a_n$. Let

$$L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

(a) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $L = 1$, then this test is inconclusive.

The proof of the Root Test is similar to that of the Ratio Test.

Example. Use the Root Test to determine if $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{3n}$ is absolutely convergent or divergent.

Solution. Here $a_n = (-1)^{3n}\left(\frac{2n}{n+1}\right)^{3n}$. Then $|a_n| = \left(\frac{2n}{n+1}\right)^{3n}$ and

$$\begin{aligned} L &= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left[\left(\frac{2n}{n+1}\right)^{3n}\right]^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^{3} \\ &= \left(\lim_{n\to\infty} \frac{2}{1+\frac{1}{n}}\right)^{3} \\ &= 2^3 \\ &= 8. \end{aligned}$$

Since $L = 8 > 1$, $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{3n}$ is divergent by the Root Test.

Note that this problem can also be done by the Ratio Test but the calculation will be a little bit more complicated:
Here $a_n = (-2)^{3n}\left(\frac{n}{n+1}\right)^{3n}$. Then $a_{n+1} = (-2)^{3n+3}\left(\frac{n+1}{n+2}\right)^{3n+3}$ and

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|(-2)^{3n+3}\left(\frac{n+1}{n+2}\right)^{3n+3}\frac{1}{(-2)^{3n}}\left(\frac{n+1}{n}\right)^{3n}\right| \\
&= \lim_{n\to\infty}\left|-8\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^{3n+3}\left(1+\frac{1}{n}\right)^{3n}\right| \\
&= \lim_{n\to\infty} 8\frac{\left(1+\frac{1}{n}\right)^3\left[\left(1+\frac{1}{n}\right)^n\right]^3}{\left(1+\frac{2}{n}\right)^3\left[\left(1+\frac{2}{n}\right)^n\right]^3}\left[\left(1+\frac{1}{n}\right)^n\right]^3 \\
&= 8\frac{1^3\cdot e^3}{1^3\cdot (e^2)^3}e^3 \quad \left(\text{since } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right) \\
&= 8.
\end{aligned}
$$

Since $L = 8 > 1$, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{3n}$ is divergent by the Ratio Test.

Exercises

1. Use the Ratio Test to determine if each of the following series is absolutely convergent or divergent:
   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n^n}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\ln(n+1)}$, (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2^n}$.

2. Use the Root Test to determine if each of the following series is absolutely convergent or divergent:
   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}$, (b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{n}\right)^{-n}$, (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos^{2n}\left(\frac{1}{n}\right)$.

Answers

1. (a) Absolutely convergent, (b) No conclusion, (c) Divergent

2. (a) Divergent, (b) Absolutely convergent, (c) No conclusion

5.6 Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

A *power series about* c (or, centered at c) is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

Example.

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a power series about 0.

2. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = 1 + \frac{(x-2)}{1!} + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots$ is a power series about 2.

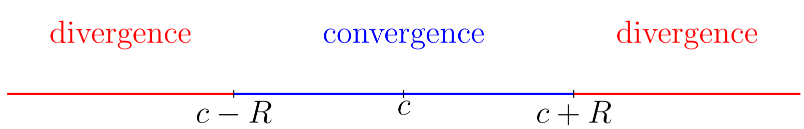
Theorem. *There are three possibilities regarding convergence of a power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$*:*

1. $\sum_{n=0}^{\infty} a_n(x-c)^n$ *converges only for* $x = c$.

2. $\sum_{n=0}^{\infty} a_n(x-c)^n$ *converges for all values of* x.

3. *There is a positive real number* R *such that* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *converges if* $|x-c| < R$ *and diverges if* $|x-c| > R$.

Example. The power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a geometric series with the common ratio x. Therefore it converges if $|x| < 1$ and diverges if $|x| \geq 1$. So in this case, $R = 1$.



Definition. If $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges for $|x-c| < R$ and diverges for $|x-c| > R$, then R is called the *radius of convergence* of $\sum_{n=0}^{\infty} a_n(x-c)^n$. We define R to be 0 or ∞ if $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges only for $x = c$ and for all values of x respectively. The *interval of convergence* of $\sum_{n=0}^{\infty} a_n(x-c)^n$ is the interval that contains all the values of x for which $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges.

Note that the interval of convergence contains $(c-R, c+R)$.

Example. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{3^n\sqrt{n}}.$$

Solution. Here

$$a_n = \frac{(2x-1)^n}{3^n\sqrt{n}} = \frac{2^n\left(x-\frac{1}{2}\right)^n}{3^n\sqrt{n}} = \left(\frac{2}{3}\right)^n \frac{\left(x-\frac{1}{2}\right)^n}{\sqrt{n}}.$$

Then

$$a_{n+1} = \left(\frac{2}{3}\right)^{n+1} \frac{\left(x-\frac{1}{2}\right)^{n+1}}{\sqrt{n+1}}$$

and

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{3}\right)^{n+1} \frac{\left(x-\frac{1}{2}\right)^{n+1}}{\sqrt{n+1}} \left(\frac{3}{2}\right)^n \frac{\sqrt{n}}{\left(x-\frac{1}{2}\right)^n} = \frac{2}{3}\left(x-\frac{1}{2}\right)\sqrt{\frac{n}{n+1}}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\left(x-\frac{1}{2}\right)\sqrt{\frac{n}{n+1}}\right| = \lim_{n\to\infty}\frac{2}{3}\left|x-\frac{1}{2}\right|\sqrt{\frac{1}{1+\frac{1}{n}}} = \frac{2}{3}\left|x-\frac{1}{2}\right|.$$

By the Ratio Test, the given power series is convergent if $\frac{2}{3}\left|x-\frac{1}{2}\right| < 1$, i.e., $\left|x-\frac{1}{2}\right| < \frac{3}{2}$ and divergent if $\frac{2}{3}\left|x-\frac{1}{2}\right| > 1$, i.e., $\left|x-\frac{1}{2}\right| > \frac{3}{2}$. Thus the radius of convergence is $R = \frac{3}{2}$.

Since $\left|x-\frac{1}{2}\right| < \frac{3}{2} \implies \frac{1}{2}-\frac{3}{2} < x < \frac{1}{2}+\frac{3}{2} \implies -1 < x < 2$, the interval of convergence contains $(-1, 2)$. Now we need to determine the convergence for $x = -1, 2$ (the endpoints).

At $x = 2$, the power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(2\cdot 2-1)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is divergent by the p-series Test with $p = \frac{1}{2} \leq 1$.

At $x = -1$, the power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(2(-1)-1)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$. Let $a_n = \frac{1}{\sqrt{n}}$. For all integers $n \geq 1$, $a_n = \frac{1}{\sqrt{n}} > 0$ and $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$. Also $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$. Thus $\{a_n\} = \{\frac{1}{\sqrt{n}}\}$ is a positive and decreasing sequence that converges to 0. By the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent.

Since the power series is convergent at -1 and divergent at 2, the interval of convergence is $[-1, 2)$.

Definition. Let I be the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$. The *sum function* f of the series is a function with domain I that is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n,$$

for all x in I.

Example. The power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ converges to $\frac{1}{1-x}$ if $|x| < 1$ and diverges if $|x| \geq 1$. So the sum function is $f(x) = \frac{1}{1-x}$ for all x in $I = (-1, 1)$ and we write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

The above example helps us write some functions as power series as illustrated by the following example.

Example. Find a power series representation of $\dfrac{2x}{x^2+9}$ and its interval of convergence.

Solution.

$$\begin{aligned}
\frac{2x}{x^2+9} = \frac{2x}{9\left(1+\frac{x^2}{9}\right)} = \frac{2x}{9}\frac{1}{\left(1-\left(-\frac{x^2}{9}\right)\right)} &= \frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n, \quad \left|-\frac{x^2}{9}\right| < 1 \\
&= \frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}, \quad |x^2| < 9 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n+1}}{9^{n+1}}, \quad |x| < 3.
\end{aligned}$$

Since the power series converges if and only if $|x| < 3$, the interval of convergence is $(-3, 3)$.

Theorem. *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *has radius of convergence* R*. Then the power series is differentiable and integrable on* $(c-R, c+R)$*. Moreover, the power series can be differentiated and integrated term by term:*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots.$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots.$$

The above can alternatively be written as follows:

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n(x-c)^n\right] = \sum_{n=0}^{\infty}\left[\frac{d}{dx}a_n(x-c)^n\right].$$

$$\int\left[\sum_{n=0}^{\infty} a_n(x-c)^n\right]dx = \sum_{n=0}^{\infty}\left[\int a_n(x-c)^n\,dx\right].$$

The preceding theorem can be used to find power series representations of some functions.

Example. Find a power series representation of each of the following functions: (a) $\ln(1+x)$, (b) $\frac{1}{(1-x)^2}$.

Solution. (a) Since $\displaystyle\int \frac{1}{1+x}\,dx = \ln(1+x)$ and

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n,\ |x|<1,$$

we have

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= \int \left[\sum_{n=0}^{\infty}(-1)^n x^n\right] dx \\ &= \sum_{n=0}^{\infty}\left[\int (-1)^n x^n\,dx\right] \\ &= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.\end{aligned}$$

To find C, we plug $x=0$:

$$\ln(1) = C + \sum_{n=0}^{\infty}(-1)^n \frac{0^{n+1}}{n+1} \implies C = 0.$$

Thus

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,\ |x|<1.$$

(b) Since $\displaystyle\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ |x|<1,$$

we have

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] \\ &= \sum_{n=0}^{\infty}\left[\frac{d}{dx}(x^n)\right] \\ &= \sum_{n=1}^{\infty} n x^{n-1},\ |x|<1.\end{aligned}$$

Exercises

1. Find the radius and interval of convergence of each of the following power series:
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n2^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n^3}$.

2. Write $\dfrac{1}{1+x^3}$ as a power series and use it to find $\displaystyle\int \frac{dx}{1+x^3}$.

3. Write $\dfrac{x}{4x+1}$ as a power series and use it to find $\displaystyle\int_0^{\frac{1}{8}} \frac{x\,dx}{4x+1}$.

4. Use the differentiation to find the power series of $\dfrac{-1}{(x+1)^2}$. $\left(\text{Hint. } \dfrac{1}{x+1}\right)$

Answers

1. (a) $R = 2,\ (1, 5]$, (b) $R = \infty,\ (-\infty, \infty)$, (c) $R = 1,\ [-2, 0]$

2. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{3n}$, $\displaystyle C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$

3. $\displaystyle\sum_{n=0}^{\infty} (-1)^n 2^{2n} x^{n+1}$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)2^{n+6}}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^{n-1}$

5.7 Taylor Series

A function f may or may not have a power series representation. But if it has one, we can find it explicitly.

Theorem. *Suppose that a function f is infinitely differentiable at c and f has a power series representation about c:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,\ |x-c| < R.$$

Then $a_n = \dfrac{f^{(n)}(c)}{n!}$, $n \geq 0$, *i.e.,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots .$$

Proof. Plugging $x = c$ in $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, we get $a_0 = f(c)$. Taking derivative of f at c, we have

$$f'(c) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1}\Big|_{x=c} \implies a_1 = f'(c).$$

Similarly taking kth derivative of f at c, we have

$$f^{(k)}(c) = \sum_{n=k}^{\infty} a_n n!(x-c)^{n-k}\Big|_{x=c} \implies f^{(k)}(c) = k!a_k \implies a_k = \frac{f^{(k)}(c)}{k!}.$$

□

Definition. The *Taylor Series* of f about c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots.$$

The *Maclaurin Series* of f is the Taylor Series of f about 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

Example.

1. Find the Taylor Series of $f(x) = e^x$ about 2.
2. Find the Maclaurin Series of $f(x) = e^x$.

Solution.

1. Since $f^{(n)}(x) = e^x$ for all integers $n \geq 0$, $f^{(n)}(2) = e^2$ for all integers $n \geq 0$. Then the Taylor Series of $f(x) = e^x$ about 2 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2 + \frac{e^2}{1!}(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots.$$

2. Since $f^{(n)}(0) = e^0 = 1$ for all integers $n \geq 0$, the Maclaurin Series of $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Note that if $f(x) = e^x$ has a power series representation about 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Question. When does a function f have a power series representation? When is $f(x)$ equal to its Taylor series?

To answer these questions, we define the *nth degree Taylor polynomial* of f about c, denoted by T_n, as

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

We define the corresponding *remainder*, denoted by R_n, as $R_n(x) = f(x) - T_n(x)$. Then

$$f(x) = T_n(x) + R_n(x).$$

Taking limit as $n \to \infty$, we have

$$f(x) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n + \lim_{n\to\infty} R_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n,$$

when $\lim_{n\to\infty} R_n(x) = 0$.

Theorem. *Suppose that a function f is infinitely differentiable at c with Taylor polynomial T_n about c and the remainder R_n. If $\lim_{n\to\infty} R_n(x) = 0$ for all x satisfying $|x-c| < R$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n,\ |x-c| < R.$$

It is difficult to show $\lim_{n\to\infty} R_n(x) = 0$. We use the following result:

Theorem. *If M is a constant such that $|f^{(n)}(x)| \leq M$ for all x satisfying $|x-c| < R$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1},\ |x-c| < R.$$

Example. Prove that $f(x) = e^x$ is equal to its Maclaurin Series for all x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Solution. Let R be an arbitrary positive real number. It suffices to show that $f(x) = e^x$ is equal to its Maclaurin Series for all x satisfying $|x| < R$.

Since $f^{(n)}(x) = e^x$ for all integers $n \geq 0$, $|f^{(n)}(x)| \leq e^R$ for all x satisfying $|x| < R$. Then by the preceding theorem,

$$|R_n(x)| \leq \frac{e^R}{(n+1)!}|x|^{n+1},\ |x| < R.$$

Taking limit as $n \to \infty$, we have

$$0 \le \lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{e^R}{(n+1)!}|x|^{n+1} = 0 \implies \lim_{n\to\infty} |R_n(x)| = 0,\ |x| < R.$$

$$f(x) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ |x| < R.$$

Since R is an arbitrary positive real number, for all x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

By arguments similar to that above, we have the following results:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,\ \text{for all } x\ (R = \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,\ \text{for all } x\ (R = \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,\ \text{for } |x| < 1\ (R = 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,\ \text{for } |x| < 1\ (R = 1)$$

Example. Write $\int e^{-x^2}\, dx$ and $\int_0^1 e^{-x^2}\, dx$ as infinite series.

Solution. We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Substituting x by $-x^2$, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots .$$

By term by term integration, we have

$$\int e^{-x^2}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\right] dx = \sum_{n=0}^{\infty} \left[\int (-1)^n \frac{x^{2n}}{n!}\, dx\right] = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

$$\int_0^1 e^{-x^2}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

Note that we can approximate $\int_0^1 e^{-x^2}\, dx$ by truncating the above series after a few terms.

$$\int_0^1 e^{-x^2}\, dx \approx \sum_{n=0}^{2} \frac{(-1)^n}{n!(2n+1)} = \frac{1}{1} - \frac{1}{3} + \frac{1}{10} = \frac{23}{30}.$$

Exercises

1. Suppose f is a function such that $f^{(n)}(0) = \frac{(n+1)!}{2^n}$ for all integers $n \geq 0$. Find the Maclaurin series of f and its radius of convergence.

2. Find the following integrals as infinite series:
(a) $\displaystyle\int \tan^{-1}(3x^2)\,dx$, (b) $\displaystyle\int_0^1 \frac{\cos x - 1}{x}\,dx$.

3. Find the first three nonzero terms in the Maclaurin series of the following functions:
(a) $e^x \sin x$, (b) $\dfrac{\ln(1+x)}{1-x}$.

Answers

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^n} x^n$, $R = 2$

2. (a) $\displaystyle C + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{4n+3}}{(2n+1)(4n+3)}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n)!}$

3. (a) $x + x^2 + \frac{x^3}{3} + \cdots$, (b) $x + \frac{x^2}{2} + \frac{5x^3}{6} + \cdots$

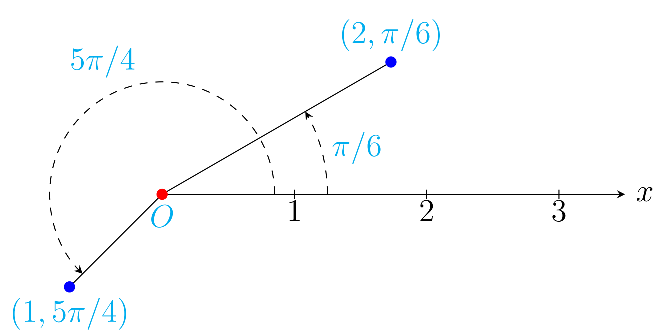
Chapter 6

Polar Coordinates and Parametric Equations

In this chapter we learn the basics of polar coordinates and parametric equations. These are useful tools for many multivariable calculus topics. In particular, integrating a function by using its parametric equations is useful in certain situations.

6.1 Polar Coordinates

We introduce a new coordinate system called the polar coordinate system which is useful in certain situations like dealing with circles or circular regions. The polar coordinate system has a fixed point called the *pole* (or the origin) denoted by O and a fixed horizontal line from the pole called the *polar axis* (or the positive x-axis). The polar coordinates of a point P in the plane are written as (r, θ) where $r \geq 0$ is the distance between P and the pole (or the origin) and θ is the angle in $[0, 2\pi)$ that is between the polar axis (or the positive x-axis) and the line joining P and the pole (or the origin). Here an angle is positive when it is measured counterclockwise. Note that $r = 0$ or $(0, \theta)$ correspond to the pole or the origin for any value of θ.

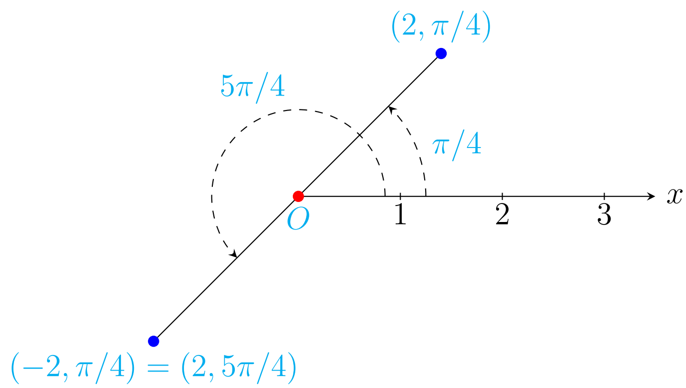


Note the following convention for polar coordinates not in the above unique representation of (r, θ):

1. $(r, \theta) = (r, \theta + 2n\pi)$ for all integers n.
2. $(-r, \theta)$ is $(r, \theta + \pi)$ when $0 \leq \theta < \pi$ and $(r, \theta - \pi)$ when $\pi \leq \theta < 2\pi$.

Example.

1. $\left(2, \frac{\pi}{4}\right) = \left(2, \frac{9\pi}{4}\right)$ as $\frac{9\pi}{4} = 2\pi + \frac{\pi}{4}$. Similarly $\left(2, \frac{\pi}{4}\right) = \left(2, -2\pi + \frac{\pi}{4}\right) = \left(2, -\frac{7\pi}{4}\right)$.
2. $\left(-2, \frac{\pi}{4}\right) = \left(2, \frac{\pi}{4} + \pi\right) = \left(2, \frac{5\pi}{4}\right)$.



Conversion between Polar and Cartesian Coordinates:
The Cartesian coordinates (x, y) of the polar coordinates (r, θ) are given by

$$x = r\cos\theta,\ \ y = r\sin\theta.$$

Note from the preceding equations that

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

The polar coordinates (r, θ) of the Cartesian coordinates (x, y) are give by

$$r = \sqrt{x^2 + y^2},\ \ \theta = \tan^{-1}\left(\frac{y}{x}\right),$$

where θ is the angle in $[0, 2\pi)$ that is between the positive x-axis and the line joining (x, y) and the origin. Note that $-\frac{\pi}{2} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{2}$ and there are two angles in $[0, 2\pi)$ that satisfy $\tan\theta = \dfrac{y}{x}$. Therefore, when $x < 0$, the correct θ is $\tan^{-1}\left(\frac{y}{x}\right) + \pi$.

Example.

1. Find the Cartesian coordinates of the following points in polar coordinates:
   (a) $(2, \pi/4)$, (b) $(4, -11\pi/6)$.

2. Find the polar coordinates (r, θ) (with $r \geq 0$, $0 \leq \theta < 2\pi$) of the following points: (a) $(\sqrt{3}, 1)$, (b) $(-2, 2)$.

Solution.

1. (a) Here $r = 2$ and $\theta = \frac{\pi}{4}$.

$$x = 2\cos\left(\frac{\pi}{4}\right) = 2\frac{\sqrt{2}}{2} = \sqrt{2},\ y = 2\sin\left(\frac{\pi}{4}\right) = 2\frac{\sqrt{2}}{2} = \sqrt{2}.$$

So the point $(2, \pi/4)$ in polar coordinates is $(\sqrt{2}, \sqrt{2})$ in Cartesian coordinates.

(b) Here $r = 4$ and $\theta = -\frac{11\pi}{6}$.

$$x = 4\cos\left(-\frac{11\pi}{6}\right) = 4\cos\left(\frac{\pi}{6} - 2\pi\right) = 4\cos\left(\frac{\pi}{6}\right) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3},$$
$$y = 4\sin\left(-\frac{11\pi}{6}\right) = 4\sin\left(\frac{\pi}{6} - 2\pi\right) = 4\sin\left(\frac{\pi}{6}\right) = 4\frac{1}{2} = 2.$$

So the point $(4, -11\pi/6)$ in polar coordinates is $(2\sqrt{3}, 2)$ in Cartesian coordinates.

2. (a) Here $x = \sqrt{3}$ and $y = 1$.

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2,\ \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

So the point $(\sqrt{3}, 1)$ in Cartesian coordinates is $(2, \frac{\pi}{6})$ in polar coordinates.

(b) Here $x = -2$ and $y = 2$.

$$r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2},\ \theta = \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

Note that $\theta = -\frac{\pi}{4}$ is not correct as it corresponds to the fourth quadrant where the point $(-2, 2)$ lies in the second quadrant. So the correct θ is $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$. Thus the point $(-2, 2)$ in Cartesian coordinates is $(2\sqrt{2}, \frac{3\pi}{4})$ in polar coordinates.

The Polar Equation of a Curve:
The equation of a curve in rectangular coordinates can be converted to a polar equation by substituting $x = r\cos\theta$, $y = r\sin\theta$. A polar equation is written in the form $r = f(\theta)$ or $F(r, \theta) = 0$.

Example.

1. The polar equation of the circle $x^2 + y^2 = a^2$, $a > 0$ is $r = a$.
   Since $x^2 + y^2 = r^2$, $r^2 = a^2 \implies r = a$.

2. The polar equation of the circle $(x-a)^2 + (y-b)^2 = a^2 + b^2$ is $r = 2a\cos\theta + 2b\sin\theta$.

$$\begin{aligned}(x-a)^2 + (y-b)^2 = a^2 + b^2 &\implies (r\cos\theta - a)^2 + (r\sin\theta - b)^2 = a^2 + b^2\\ &\implies r^2 - 2r(a\cos\theta + b\sin\theta) + a^2 + b^2 = a^2 + b^2\\ &\implies r(r - 2a\cos\theta - 2b\sin\theta) = 0\\ &\implies r = 2a\cos\theta + 2b\sin\theta.\end{aligned}$$

3. The polar equation of the line $y = mx$ is $\theta = \tan^{-1} m$ (where $-\infty < r < \infty$).

$$y = mx \implies r\sin\theta = mr\cos\theta \implies \tan\theta = m.$$

   Note that $\theta = \tan^{-1} m,\ r \geq 0$ and $\theta = \pi + \tan^{-1} m,\ r \geq 0$ represent two lines starting from the origin whose union is the line $y = mx$. If we allow r to be negative, $\theta = \pi + \tan^{-1} m,\ r \geq 0$ can be written as $\theta = \tan^{-1} m,\ r \leq 0$. Therefore, the equation simply becomes $\theta = \tan^{-1} m,\ -\infty < r < \infty$.

4. The polar equation of the line whose closest point from the origin is (d, α) in polar coordinates is $r = d\sec(\theta - \alpha)$.
   Consider an arbitrary point $P(r, \theta)$ on the line. From the right triangle formed by P, (d, α), and the origin, we have

$$\frac{d}{r} = \cos(\theta - \alpha) \implies r = d\sec(\theta - \alpha).$$

5. There are famous curves with simple polar equations such as spiral (e.g., $r = \theta$), cardioid (e.g., $r = 1 - \cos\theta$), lemniscate (e.g., $r^2 = \cos(2\theta)$), and polar rose (e.g., $r = \cos(2\theta)$).

Example. Write a Cartesian equation for the cardioid $r = 1 - \cos\theta$.

Solution. Substituting $r = \sqrt{x^2 + y^2}$ and $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$ in $r = 1 - \cos\theta$, we have

$$\begin{aligned}\sqrt{x^2 + y^2} = 1 - \frac{x}{\sqrt{x^2 + y^2}} &\implies x^2 + y^2 = \sqrt{x^2 + y^2} - x\\ &\implies x^2 + y^2 + x = \sqrt{x^2 + y^2}\\ &\implies (x^2 + y^2 + x)^2 = x^2 + y^2.\end{aligned}$$

Exercises

1. Find the Cartesian coordinates of the following points in polar coordinates:
   (a) $(2, \pi/6)$, (b) $(2, 13\pi/6)$.

2. Find the polar coordinates of the following points:
   (a) $(-1, \sqrt{3})$, (b) $(1, -1)$. (Use $r \geq 0$, $0 \leq \theta < 2\pi$)

3. Find a polar equation in the form $r = f(\theta)$ for each of the following curves:
   (a) $(x-1)^2 + (y+2)^2 = 5$, (b) $x^2 + 4y^2 = 9$.

4. Find the Cartesian equations of the following polar curves:
   (a) $r = \dfrac{3}{2\cos\theta - \sin\theta}$, (b) $r = \cos(2\theta)$.

Answers

1. (a) $(\sqrt{3}, 1)$, (b) $(\sqrt{3}, 1)$
2. (a) $(2, 2\pi/3)$, (b) $(\sqrt{2}, 7\pi/4)$
3. (a) $r = 2\cos\theta - 4\sin\theta$, (b) $r = \frac{3}{\sqrt{\cos^2\theta + 4\sin^2\theta}}$
4. (a) $y = 2x - 3$, (b) $(x^2 + y^2)^{3/2} = x^2 - y^2$

6.2 Parametric Equations

Sometimes it is impossible to write the equation of a curve in the form $y = f(x)$ or $x = f(y)$. But we may be able to write x and y in terms of a third variable or parameter, say t, in a set of equations called *parametric equations*:

$$x = f(t),\ y = g(t),\ a \leq t \leq b.$$

A curve given by parametric equations is called a *parametric curve* which is sometimes written as

$$c(t) = (f(t), g(t)),\ a \leq t \leq b.$$

Note that $c(t)$ is drawn with an orientation using the values of t, starting from the initial point $(f(a), g(a))$ and ending at the terminal point $(f(b), g(b))$.

Example.

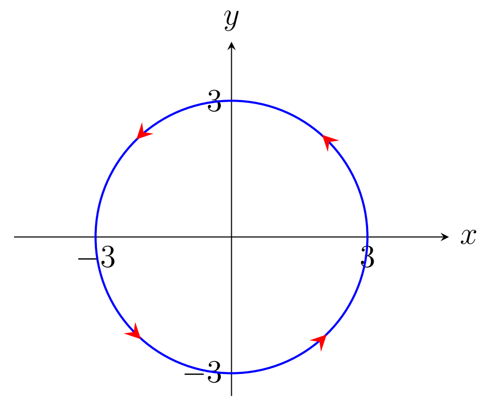
1. The parametric equations of the circle $x^2 + y^2 = 9$ are

   $$x = 3\cos t,\ y = 3\sin t,\ 0 \leq t \leq 2\pi.$$

   It can be verified as follows:

   $$x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9.$$

   The top and bottom halves of the circle are given by $0 \leq t \leq \pi$ and $\pi \leq t \leq 2\pi$ respectively.

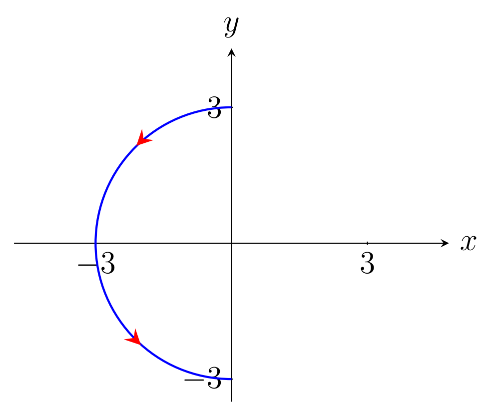


$x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$

The left semicircle of the preceding circle can be written as

$$x = 3\cos t,\ y = 3\sin t,\ \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$$

The initial point of the semicircle is $\left(3\cos\left(\frac{\pi}{2}\right), 3\sin\left(\frac{\pi}{2}\right)\right) = (0, 3)$ and the terminal point is $\left(3\cos\left(\frac{3\pi}{2}\right), 3\sin\left(\frac{3\pi}{2}\right)\right) = (0, -3)$.



$x = 3\cos t,\ y = 3\sin t,\ \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

2. The parametric equations of the circle $(x-a)^2 + (y-b)^2 = r^2$ are

$$x = a + r\cos t,\ y = b + r\sin t,\ 0 \le t \le 2\pi.$$

3. The parametric equations of the ellipse $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{b^2} = 1$ are

$$x = \alpha + a\cos t,\ y = \beta + b\sin t,\ 0 \le t \le 2\pi.$$

4. The parametric equations of the line segment from (a_1, b_1) to (a_2, b_2) are

$$x = (1-t)a_1 + ta_2,\ y = (1-t)b_1 + tb_2,\ 0 \le t \le 1.$$

5. The parametric equations of the line $y = mx + b$ are

$$x = t,\ y = mt + b,\ -\infty < t < \infty.$$

The above is a natural parametrization of the line where x is the parameter t and y is a function of t.

Implicitization of Parametric Curves:
Sometimes we can eliminate the parameter t form parametric equations $x = f(t),\ y = g(t)$ of a curve and write an implicit equation in x and y.

Example. Find an equation for each of the following parametric curves:
(a) $x = t^2 + 2$, $y = t + 1$, $-1 \le t < \infty$, (b) $c(t) = (\tan t, \sec^2 t)$, $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Solution. (a) By solving for t from $y = t + 1$, we get $t = y - 1$. Substituting $t = y - 1$ in $x = t^2 + 2$, we get

$$x = (y - 1)^2 + 2.$$

Note that $t = -1$ corresponds to $(3, 0)$ and $y = t + 1$ increases with t. Thus the above parabola starts from $(3, 0)$ and goes to the increasing direction of y.

(b) Here $x = \tan t$ and $y = \sec^2 t$. Since $\sec^2 t - \tan^2 t = 1$,

$$y - x^2 = 1 \implies y = x^2 + 1.$$

Since $-\infty < \tan t < \infty$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$, $-\infty < x < \infty$. Thus $c(t) = (\tan t, \sec^2 t)$, $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$ represents the parabola $y = x^2 + 1$.
Note that the parabola $y = x^2 + 1$ can be parametrized in multiple ways. For example, its natural parametrization is

$$c(t) = (t, t^2 + 1),\ -\infty < t < \infty.$$

Derivatives from Parametric Equations:
Consider the following parametric equations:

$$x = f(t),\ y = g(t),\ a \le t \le b,$$

where x and y are differentiable functions of t and y is a differentiable function of x. Then y is a differentiable function of t. By the chain rule,

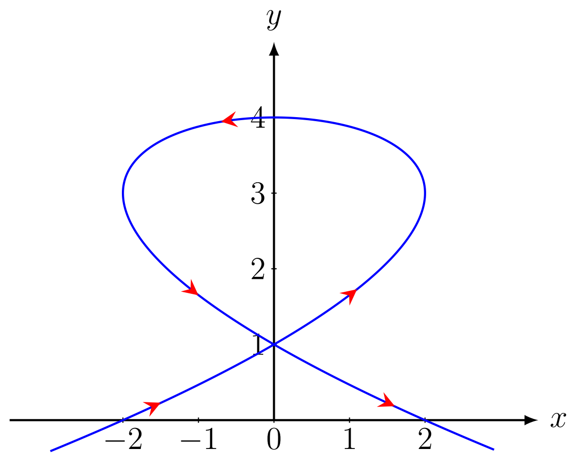
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

which implies

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)},\ \text{ provided } x'(t) \ne 0.$$

Example. Find the equations of the tangent lines to the parametric curve $c(t) = (t^3 - 3t, 4 - t^2)$ at $(0, 1)$.

Solution. First note that $t^3 - 3t = t(t^2 - 3) = 0 \implies t = 0, \pm\sqrt{3}$. Since $t = 0 \implies y = 4 - t^2 = 4$ and $t = \pm\sqrt{3} \implies y = 4 - t^2 = 1$, both $t = \sqrt{3}$ and $t = -\sqrt{3}$ correspond to $(0, 1)$ where $c(t) = (t^3 - 3t, 4 - t^2)$ self-intersects.



$c(t) = (t^3 - 3t, 4 - t^2)$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-2t}{3t^2 - 3}.$$

The slope of the tangent line corresponding to $t = \sqrt{3}$ is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \left.\frac{-2t}{3(t^2 - 1)}\right|_{t=\sqrt{3}} = \frac{-\sqrt{3}}{3}.$$

Thus the equation of the tangent line corresponding to $t = \sqrt{3}$ is

$$y - 1 = \frac{-\sqrt{3}}{3}x.$$

Similarly the slope of the tangent line corresponding to $t = -\sqrt{3}$ is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \left.\frac{-2t}{3(t^2 - 1)}\right|_{t=-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

and the equation of the tangent line corresponding to $t = -\sqrt{3}$ is

$$y - 1 = \frac{\sqrt{3}}{3}x.$$

Integration from Parametric Equations:
Suppose that y is an integrable function of x on $[a, b]$ given by the following parametric equations:

$$x = f(t),\ y = g(t),\ \alpha \leq t \leq \beta,$$

where $a = f(\alpha)$, $b = f(\beta)$, and x is a differentiable function of t. Since $dx = f'(t)\ dt$,

$$\int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt = \int_\alpha^\beta y(t)x'(t)\ dt.$$

Example. Find the area of the region enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. Suppose that the equation of the top half of the ellipse is $y = f(x)$, $-a \leq x \leq a$. Then $f(x) \geq 0$ for $-a \leq x \leq a$. By the symmetry of the ellipse, the required area is

$$2\int_{-a}^{a} f(x)\ dx.$$

Now $y = f(x)$, $-a \leq x \leq a$ is parametrized as

$$x = a\cos t,\ y = b\sin t,$$

where t goes from $t = \pi$ to $t = 0$ so that $-a = a\cos(\pi)$ and $a = a\cos(0)$. Therefore, the required area is

$$\begin{aligned}
2\int_{-a}^{a} f(x)\ dx &= 2\int_\pi^0 y(t)x'(t)\ dt \\
&= 2\int_\pi^0 b\sin t(-a\sin t)\ dt \\
&= -ab\int_\pi^0 2\sin^2 t\ dt \\
&= ab\int_0^\pi (1-\cos(2t))\ dt \\
&= ab\left(t - \frac{\sin(2t)}{2}\right)\Bigg|_0^\pi \\
&= \pi ab.
\end{aligned}$$

In case you do not like t going backward from $t = \pi$ to $t = 0$ in the above parametrization of the top half of the ellipse, you may use the following alternative parametrization:

$$x = a\cos(\pi - t),\ y = b\sin(\pi - t),\ 0 \leq t \leq \pi.$$

Then

$$2\int_{-a}^{a} f(x)\ dx = 2\int_0^\pi y(t)x'(t)\ dt = 2\int_0^\pi b\sin(\pi - t)a\sin(\pi - t)\ dt = \pi ab.$$

Exercises

1. Find the parametric equations of the following curves:
   (a) The line with slope 2 passing through $(-1, 2)$.
   (b) The line segment from $(-1, 2)$ to $(4, 3)$.
   (c) The circle with radius 2 centered at $(-1, 2)$.
   (d) $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1$
2. Find an equation in the form $y = f(x)$ for each of the following parametric curves:
   (a) $x = 1 + t^{-1}$, $y = 2t^3$, $t \neq 0$
   (b) $x = 1 + e^t$, $y = \ln(1 + t)$, $-1 < t < \infty$
   (c) $x = \sin t$, $y = \tan t$, $-\frac{\pi}{2} < t < 0$
3. Consider the parametric curve $c(t) = (t^2 - 9, 4t - t^3)$.
   (a) Show that the curve self-intersects at $(-5, 0)$.
   (b) Find the tangent line at $t = 1$.
   (c) A sketch of the curve shows that it encloses a region by self-intersecting at $(-5, 0)$. Find the area of the region.

Answers

1. (a) $x = t,\ y = 2t + 4,\ -\infty < t < \infty$, (b) $x = 5t - 1,\ y = t + 2,\ 0 \leq t \leq 1$, (c) $x = -1 + 2\cos t,\ y = 2 + 2\sin t,\ 0 \leq t < 2\pi$, (d) $x = 1 + 2\cos t,\ y = -2 + 3\sin t$, $0 \leq t < 2\pi$
2. (a) $y = \dfrac{2}{(x-1)^3}$, (b) $y = \ln(1 + \ln(x - 1))$, $x > 1 + e^{-1}$, (c) $y = \dfrac{x}{\sqrt{1 - x^2}}$, $-1 < x < 0$
3. (a) $t = \pm 2$ both give $(-5, 0)$, (b) $y - 3 = \frac{1}{2}(x + 8)$, (c) $2\displaystyle\int_0^2 y(t)x'(t)\,dt = \frac{256}{15}$

6.3 Arc Length and Speed

Consider a curve given by

$$x = f(t),\ y = g(t),\ a \leq t \leq b,$$

where the curve is traversed exactly once for t varying from a to b. If f' and g' are continuous on $[a, b]$, then the arc length L of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Note that if the curve is $y = f(x)$ on $[a, b]$, then its natural parametrization is

$$x = t,\ y = f(t),\ a \le t \le b,$$

and consequently the above arc length formula becomes that in section 3.6:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example. Find the circumference of a circle of radius r.

Solution. Consider the following circle of radius r:

$$x = r\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi.$$

Here $\dfrac{dx}{dt} = -r\sin t$ and $\dfrac{dy}{dt} = r\cos t$. Then the circumference L is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{2\pi} r\, dt \\ &= rt\Big|_0^{2\pi} \\ &= 2\pi r. \end{aligned}$$

Note that if we consider the following parametrization of the circle

$$x = r\cos t,\ y = r\sin t,\ 0 \le t \le 4\pi,$$

where the circle is traversed twice, then the arc length formula gives twice the circumference:

$$\int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{4\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = 4\pi r.$$

Therefore, the arc length formula gives us the arc length of a curve from its parametrization for which the curve is traversed exactly once.

Example. Find the arc length of one arch of the cycloid:

$$x = r(t - \sin t),\ y = r(1 - \cos t),$$

which is traced by a fixed point on a circle of radius r rolling on the x-axis.

Solution. Here $\frac{dx}{dt} = r(1 - \cos t)$ and $\frac{dy}{dt} = r \sin t$. Since one arch of the cycloid is traced by varying t from 0 to 2π, its arc length L is

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(r(1 - \cos t))^2 + (r \sin t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt \\
&= \int_0^{2\pi} r\sqrt{2(1 - \cos t)}\, dt \\
&= \int_0^{2\pi} r\sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt \\
&= \int_0^{2\pi} 2r \sin\left(\frac{t}{2}\right) dt \quad \left(\text{since } \sin\left(\frac{t}{2}\right) \geq 0 \text{ for } 0 \leq t \leq 2\pi\right) \\
&= -4r\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} \\
&= -4r\cos(\pi) + 4r\cos(0) \\
&= 8r.
\end{aligned}$$

Suppose that a particle is moving in the xy-plane along a parametric curve $c(t) = (x(t), y(t))$ where time t starts from $t = a$. Then the distance $s(t)$ traveled over the time interval $[a, t]$ is

$$s(t) = \int_a^t \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Since the speed is the rate of change of distance traveled with respect to time, the speed at time $t = c$ is

$$s'(c) = \left(\frac{d}{dt}\int_a^t \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt\right)\Big|_{t=c} = \sqrt{[x'(t)]^2 + [y'(t)]^2}\Big|_{t=c}.$$

Example. Suppose that a particle is moving along the curve

$$x = e^t + e^{-t},\ y = 2t,\ 0 \leq t < \infty.$$

(a) Find the speed (in m/s) at $t = 2$ sec.

(b) Find the distance traveled during the time interval $[0, 3]$.

Solution. Here $x'(t) = e^t - e^{-t}$ and $y'(t) = 2$.

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{(e^t - e^{-t})^2 + 2^2} = \sqrt{e^{2t} + e^{-2t} + 2} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

(a) The speed at $t = 2$ sec is

$$s'(2) = \sqrt{[x'(t)]^2 + [y'(t)]^2}\Big|_{t=2} = e^t + e^{-t}\Big|_{t=2} = e^2 + e^{-2} \text{ m/s}.$$

(b) The distance traveled during the time interval $[0, 3]$ is

$$\begin{aligned} s(3) &= \int_0^3 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \\ &= \int_0^3 (e^t + e^{-t})\, dt \\ &= (e^t - e^{-t})\Big|_0^3 \\ &= e^3 - e^{-3} \text{ m}. \end{aligned}$$

Exercises

1. Find the arc length of the following parametric curves:

   (a) $x = \dfrac{t^2}{2},\ y = \dfrac{t^4}{4},\ 0 \le t \le 1$

   (b) $c(t) = (t, \ln t),\ 1 \le t \le 2.$

2. Suppose that a particle is moving on the spiral $x = t\cos t,\ y = t\sin t,\ 0 \le t < \infty$.

   (a) Find the speed (in m/s) at $t = 2$ sec.

   (b) Find the distance traveled during the time interval $[0, 2\pi]$.

Answers

1. (a) $\dfrac{\sqrt{2} + \ln(1+\sqrt{2})}{4}$, (b) $\sqrt{5} + \ln\left(\dfrac{\sqrt{5}-1}{2}\right) - \sqrt{2} - \ln\left(\sqrt{2}-1\right) \approx 1.22$

2. (a) $\sqrt{5}$ m/s, (b) $\pi\sqrt{1+4\pi^2} + \frac{\ln\left(2\pi+\sqrt{1+4\pi^2}\right)}{2} \approx 21.25\ m$

Bibliography

[1] Robert G. Bartle and Donald R. Sherbert, *Introduction to Real Analysis*, Wiley, 2011.

[2] Jon Rogawski and Colin Adams, *Calculus Early Transcendentals*, W. H. Freeman, 2014.

[3] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.

[4] James Stewart, *Calculus: Early Transcendentals*, Cengage Learning, 2015.

Subject Index